Frequency response of cantilever beams immersed in viscous fluids

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1 Introduction

The dynamic response of microcantilever beams is used in a broad range of applications, including ultra-sensitive mass measurements and imaging with molecular and atomic scale resolution (Newell, 1968; Binnig et al., 1986; Berger et al., 1997; Craighead, 2000; Sader, 2002; Lavrik et al., 2004; Ekinci & Roukes, 2005). Importantly, many of these applications are performed in a fluid environment (gas or liquid), which can significantly affect the dynamic response of a microcantilever. To explain this behavior, theoretical models have been developed that rigorously account for the effect of the surrounding fluid, which have been validated by detailed experimental measurements; models exist for flexural, torsional and extensional vibrational modes of cantilever beams (Sader, 1998; Green & Sader, 2002; Paul & Cross, 2004; Maali et al., 2005; Clarke et al., 2005; Basak et al., 2006; Paul et al., 2006; van Eysden & Sader, 2007, 2009b; Brumley et al., 2010; Castille et al., 2010; Cox et al., 2012). These studies establish that viscosity plays an essential role in the frequency response of cantilevers of microscopic size (∼100µm in length), such as those used in the atomic force microscope (AFM) and in micro-electromechanical systems (MEMS). This contrasts to macro-scale cantilevers (∼1m in length) which are insensitive to the effects of fluid viscosity (Chu, 1963; Lindholm et al., 1965).

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The most common approach used to model the frequency response of a cantilever beam immersed in a viscous fluid is to approximate the fluid motion by a two-dimensional flow field due to local displacement of the beam (Sader, 1998). Significantly, for cantilever beams of high aspect ratio (length/width), such as those used in the AFM, this approach has proven to be highly successful in predicting the frequency response of the fundamental mode and the next few higher order modes. Of recent interest has been the use of higher order cantilever modes (Maali et al., 2005; Braun et al., 2005; Ghatkesar et al., 2008), which enables operation at high quality factors in highly viscous fluids. However, as mode order increases, the spatial wavelength of the modes decreases, ultimately leading to violation of the two-dimensional flow field approximation and breakdown in the validity of these models. This limitation is also present in the well-known theory of Chu (1963) for the resonant frequencies of a cantilever beam immersed in an inviscid fluid, which is valid for cases where fluid viscosity can be ignored. Models that account for three-dimensional effects, and thus are valid for all modes, have been developed for both inviscid and viscous fluids (Elmer & Dreier, 1997; van Eysden & Sader, 2006a, 2007).

Operation at higher mode numbers not only increases the importance of (incompressible) three-dimensional effects discussed above, but fluid compressibility can also play a major role. The reason for this feature is that the acoustic wavelength in the fluid decreases with increasing mode number and ultimately becomes comparable to and smaller than the dominant hydrodynamic length scale of the beam. Models that rigorously account for the three-dimensional effects of fluid compressibility are reported in (van Eysden & Sader, 2009b).

In this chapter, we present an overview of the above-mentioned theoretical models for the fluid-structure interaction of cantilever beams immersed in fluid. The focus is on beams whose lengths greatly exceed their widths, as is often encountered in practice. In §2.1, the two-dimensional theory for flexural modes at low mode number is presented, where fluid viscosity plays an essential role. Models for torsional, in-plane and extensional modes, also at low mode numbers, are summarized in §2.2, §2.3, §2.4, respectively. The general, three-dimensional theory for arbitrary mode order is presented in §3, with a focus on the flexural modes; models for torsional modes are reported in (van Eysden & Sader, 2007, 2009b) and not detailed here. Both incompressible (§3.1) and compressible (§3.2) flows are discussed, accompanied by scaling analyses for determining the regimes where fluid compressibility is expected to be important.

2 Low order modes

2.1 Flexural oscillation

In many applications, such as those found in AFM and MEMS, the fundamental mode and the next few modes are typically interrogated. In such cases, flow around the cantilever can be well approximated by one that is two-dimensional and explicit formulas for the frequency response derived. The theory pertaining
A rectangular cantilever beam immersed in a viscous fluid was first derived by Sader (1998). This model is summarised in this section, and contains the following principal assumptions:

1. Cross section of the beam is uniform along its entire length and is rectangular in geometry;
2. Length of the beam $L$ greatly exceeds its width $b$, which in turn greatly exceeds its thickness $h$, see Figure 1;
3. Amplitude of oscillation is far smaller than any geometric length scale of the beam, which allows the Navier-Stokes equations to be linearised;
4. Internal (structural) dissipative effects are negligible in comparison with those of the fluid; and
5. Fluid is incompressible in nature and is unbounded in space.

The governing equation for the elastic deformation of the beam executing flexural oscillations is (Landau & Lifshitz, 1970)

$$EI \frac{\partial^4 w(x, t)}{\partial x^4} + \mu \frac{\partial^2 w(x, t)}{\partial t^2} = F(x, t),$$

where $w(x, t)$ is the deflection function of the beam in the $z$-direction, $E$ is Young’s modulus, $I$ is the second moment of area (often referred to as the moment of inertia) of the beam cross section, $\mu$ is the mass per unit length of the beam, $F(x, t)$ is the external applied force per unit length in the $z$-direction, $x$ is the spatial coordinate along the length of the beam, and $t$ is time; see Figure 1. For a cantilever beam of uniform rectangular cross section, $I = bh^3/12$ (Landau & Lifshitz, 1970) and $\mu = \rho_c bh$, where $\rho_c$ is the cantilever density. The corresponding boundary conditions for the (clamped-free) cantilever beam
are

\[
\begin{bmatrix}
w(x, t) = \frac{\partial w(x, t)}{\partial x} \\
\end{bmatrix}
\bigg|_{x=0}
= \begin{bmatrix}
\frac{\partial^2 w(x, t)}{\partial x^2} \\
\frac{\partial^3 w(x, t)}{\partial x^3} \\
\end{bmatrix}
\bigg|_{x=L} = 0.
\] 

(2)

To calculate the frequency response of the beam, we take the Fourier transform of the governing equation (1). Scaling the spatial variable, \( x \), by the length \( L \) gives

\[
\frac{EI}{L^4} \frac{\partial^4 \tilde{w}(x|\omega)}{\partial x^4} - \mu \omega^2 \tilde{w}(x|\omega) = \tilde{F}(x|\omega),
\]

(3)

where \( x \) now denotes the scaled length coordinate, and

\[
\tilde{X}(\omega) = \int_{-\infty}^{\infty} X(t)e^{i\omega t} dt,
\]

(4)

is the Fourier transform of any function \( X \) of time and \( i \) is the usual imaginary unit. For simplicity, we shall henceforth omit the superfluous “\( \sim \)” notation in (4), while noting that all dependent variables refer to their Fourier space counterparts. Solving (3) in the absence of any applied load, leads to the well-known result for the vacuum radial resonant frequencies

\[
\omega_{\text{vac}, n} = \frac{C_n^2}{L^2} \sqrt{\frac{EI}{\mu}},
\]

(5)

where \( n = 1, 2, ... \) is the mode order and \( C_n \) is the \( n \)th positive root of

\[
1 + \cos C_n \cosh C_n = 0,
\]

(6)

which is well approximated by \( C_n \approx (n - 1/2)\pi \) for \( n \geq 2 \).

We note in passing that while the present analysis focuses on cantilever beams, it is equally applicable to clamped-clamped beams provided (i) the eigenvalues \( C_n \) for clamped-clamped beams are used, and (ii) the deflection function \( w(x, t) \), and corresponding homogeneous solutions \( \phi_n(x) \), for a clamped-clamped beam are implemented in subsequent analyses. These are obtained by applying the appropriate boundary conditions for a clamped-clamped beam in (1). This yields the eigenvalues \( C_n \) satisfying \(-1 + \cos C_n \cosh C_n = 0\), which are well approximately by \( C_n \approx (n + 1/2)\pi \) for all mode numbers \( n \geq 1 \) (Landau & Lifshitz, 1970). This feature also holds for other theoretical models, including Chon et al. (2000); Green & Sader (2002, 2005); van Eysden & Sader (2006a, 2007, 2009a).

Next, we turn our attention to the analysis of the load \( F(x|\omega) \) applied to the cantilever. This load is decomposed into two components:

\[
F(x|\omega) = F_{\text{hydro}}(x|\omega) + F_{\text{drive}}(x|\omega),
\]

(7)

corresponding to the hydrodynamic load \( F_{\text{hydro}}(x|\omega) \) exerted by the fluid on the cantilever beam due to its motion, and a driving force \( F_{\text{drive}}(x|\omega) \) that excites the cantilever. The hydrodynamic load is expressed generally as

\[
F_{\text{hydro}}(x|\omega) = \frac{\pi}{4} \rho \omega^2 b^2 \Gamma f(\omega) w(x|\omega),
\]

(8)
where $\rho$ is the fluid density and $\Gamma^f(\omega)$ is the normalised hydrodynamic load, termed the hydrodynamic function. The superscript $f$ refers to the flexural modes and will be used to distinguish between results for the torsional modes. It is important to emphasise that (8) is derived under the assumption that the length of the cantilever greatly exceeds its width, and as such, is formally consistent with the underlying assumptions of the beam equation, (1). The hydrodynamic function $\Gamma^f(\omega)$ is dimensionless and depends on the radial frequency $\omega$ through the dimensionless parameter

$$ \text{Re} = \frac{\rho \omega b^2}{\eta}, $$

where $\eta$ is the fluid shear viscosity. The parameter $\text{Re}$ is commonly termed the Reynolds number\(^1\) and indicates the importance of inertial forces in the fluid relative to viscous forces.

Cantilevers used in the AFM and MEMS applications are small enough to be significantly excited by the thermal motion of molecules in the fluid in which they are immersed; they undergo Brownian motion, in accordance with the fluctuation-dissipation theorem (Landau & Lifshitz, 1969). Equipartition of energy dictates that an energy of $k_B T/2$ is imparted to each mode of oscillation, where $k_B$ is Boltzmann’s constant and $T$ is the absolute temperature. For the case where Brownian motion of the fluid molecules excite the cantilever, the thermal noise spectrum of the squared magnitude of the displacement function is (van Eysden & Sader, 2008, 2009b)

$$ |w(x|\omega)|^2_s = \frac{3\pi k_B T}{2k} \frac{\rho b}{\rho_c h} \frac{C^4_i}{\omega_{\text{vac},1}^2} \sum_{n=1}^{\infty} \frac{\omega \Gamma^f_i(\omega)}{|C^4_n - B^4_n(\omega)|^2} \phi^2_n(x). $$

The corresponding result for the slope is

$$ \left| \frac{\partial w(x|\omega)}{\partial x} \right|^2_s = \frac{3\pi k_B T}{2k} \frac{\rho b}{\rho_c h} \frac{C^4_i}{\omega_{\text{vac},1}^2} \sum_{n=1}^{\infty} \frac{\omega \Gamma^f_i(\omega)}{|C^4_n - B^4_n(\omega)|^2} \left( \frac{d\phi_n(x)}{dx} \right)^2, $$

where the subscript $s$ refers to the spectral density, $k$ is the normal spring constant of the cantilever beam (Roark, 1943), the subscript $i$ refers to the imaginary component, and the function $B_n(\omega)$ is defined as

$$ B_n(\omega) = C_1 \left( \frac{\omega}{\omega_{\text{vac},1}} \right)^{1/2} \left( 1 + \frac{\pi \rho b}{4 \rho_c h} \Gamma^f(\omega) \right)^{1/4}. $$

In the limit of small dissipative effects, i.e., when the quality factor greatly exceeds unity, the modes of the cantilever are approximately uncoupled and the frequency response of each mode of the beam is well approximated by that of a simple harmonic oscillator (SHO). We obtain the following expressions for the

\(^1\)The convention adopted for the Reynolds number conforms with Batchelor (1974). The Reynolds number is often associated with the nonlinear convective inertial term in the Navier-Stokes equation. This latter convention has not been adopted here.
resonant frequency $\omega^f_R$ (defined as the undamped in-fluid natural frequency) and quality factor $Q^f_n$ for the $n^{th}$ mode of vibration (Sader, 1998):

$$\frac{\omega_{R,n}}{\omega_{vac,n}} = \left(1 + \frac{\pi \rho b}{4 \rho_c h} \Gamma^f_f(\omega_{R,n})\right)^{-1/2}, \quad (13)$$

$$Q_n = \frac{4 \rho_c h}{\pi \rho b} + \frac{\Gamma^f_f(\omega_{R,n})}{\Gamma^f_f(\omega_{R,n})}, \quad (14)$$

where the subscript $r$ refers to the real component. Note that for a driven cantilever, the resonant frequency corresponds to the frequency at which the driving force is $90^\circ$ out of phase with the displacement of the cantilever; in general this does not coincide with the frequency of maximum amplitude of each resonance peak. We emphasize that the expressions in (13) and (14), which are formally derived in the limit of large quality factor, are also valid in the Stokes limit ($Re \to 0$), where the added apparent mass and damping coefficient of the cantilever are frequency independent.

To use the results (10)–(14), the hydrodynamic function, $\Gamma^f_f(\omega)$, must be determined. This is obtained from solution of the linearised Navier Stokes equation for a rigid beam of infinite length, with identical cross section to that of the cantilever beam, undergoing transverse oscillatory motion. For a cantilever of thin rectangular cross section, this is well approximated by that of an infinitely thin blade:

$$\Gamma^f_{rect}(\omega) = \Omega(\omega) \Gamma^f_{circ}(\omega), \quad (15)$$

where the real and imaginary parts of the correction $\Omega(\omega)$ are given by

$$\Omega_r(\omega) = \begin{align*}
(0.91324 - 0.48274 \tau + 0.46842 \tau^2 - 0.12886 \tau^3 \\
+0.044055 \tau^4 - 0.0035117 \tau^5 + 0.00069085 \tau^6) \\
\times (1 - 0.56964 \tau + 0.48690 \tau^2 - 0.13444 \tau^3 + 0.045155 \tau^4 \\
-0.0035862 \tau^5 + 0.00069085 \tau^6\right)^{-1},
\end{align*}$$

$$\Omega_i(\omega) = \begin{align*}
(-0.024134 - 0.029256 \tau + 0.016294 \tau^2 - 0.00010961 \tau^3 \\
+0.000064577 \tau^4 - 0.000044510 \tau^5) \times (1 - 0.59702 \tau \\
+0.55182 \tau^2 - 0.18357 \tau^3 + 0.079156 \tau^4 - 0.014369 \tau^5 \\
+0.0028361 \tau^6)^{-1},
\end{align*}$$

$$\tau = \log_{10}(Re/4), \quad (16)$$

and $\Gamma_{circ}(\omega)$ is the hydrodynamic function for a circular cylinder:

$$\Gamma^f_{circ}(\omega) = 1 + \frac{4i K_1 \left(-i \sqrt{iRe/4}\right)}{\sqrt{iRe/4} K_0 \left(-i \sqrt{iRe/4}\right)}, \quad (17)$$
where \( K_n(x) \) are modified Bessel functions of the second kind (Abramowitz & Stegun, 1972), and the Reynolds number is defined in (9). Equation (15) is accurate to within 0.1% over the entire range \( 0.1 < \text{Re} < 1000 \) for both real and imaginary parts, and possesses the correct asymptotic forms \( \text{Re} \to 0 \) and \( \text{Re} \to \infty \). In the absence of any simple exact analytical formula for the hydrodynamic function of an infinitely thin rectangular beam, (15) serves as a useful, practical expression.

Using the expression for the hydrodynamic function (15), and the above theoretical model, the frequency response of the cantilever can be explored. The frequency response is completely characterised by two dimensionless parameters:

\[
\text{Re} = \frac{\rho \omega_{\text{vac},1} b^2}{\eta}, \quad \bar{T} = \frac{\rho b}{\rho_c h},
\]

which enable us to study cantilevers of arbitrary dimensions and material properties, immersed in fluids of arbitrary density and viscosity. The first parameter \( \text{Re} \) is a normalised Reynolds number, which indicates the relative importance of inertial forces in the fluid to viscous forces. If this parameter greatly exceeds unity, then viscous forces exert a relatively weak effect and the flow will be predominantly inviscid in nature. The second parameter \( \bar{T} \) is the ratio of the added-apparent mass due to inertial forces in the fluid (in the absence of viscous effects), to the total mass of the beam. This parameter indicates the relative strength of fluid inertia and is used to distinguish between immersion in gas and liquid, as will be discussed below.

Note that values of \( \bar{T} \) for beams immersed in gases and liquids typically differ by three orders of magnitude. This is a direct result of the difference in densities of gases relative to those of liquids. In contrast, values for \( \text{Re} \) in gases and liquids differ by only one order of magnitude, since the kinematic viscosities of gases are typically one order of magnitude greater than those of liquids. Consequently, we shall present separate results for gases and liquids that account for these differences.

In Figure 2 we present results for the peak frequency \( \omega_p \) (i.e., that causing the peak response) and quality factor \( Q \) of the fundamental mode, as a function of both \( \text{Re} \) and \( \bar{T} \). The peak frequency is numerically calculated from the frequency response, (13), whereas the quality factor is obtained directly from (14). Consequently, results presented for \( Q \) give quantitative information about the resonance peak provided \( Q \gg 1 \), since the analogy with the frequency response of a SHO is only valid in those cases. For \( Q \leq O(1) \), however, no such analogy exists and \( Q \) only presents qualitative information about the resonance peak. In particular, for \( Q \leq O(1) \) one can conclude that substantial broadening of the resonance peak is present, and that the modes are significantly coupled in the frequency domain. A reduction in \( Q \) will then result in further broadening of the peak and an increased coupling of the modes. Finally, it is interesting to note that the definition of \( \text{Re} \) in (9) differs from that used in Sader (1998) by a factor of 4. This is because (9) is the form that appears naturally in the exact solution to \( \Gamma_f(\omega) \) around an infinitely flat blade; see van Eysden & Sader (2006b, 2009a). For consistency, this definition is adopted throughout this chapter.
Figure 2: Peak frequency \( \omega_p \) of fundamental resonance relative to frequency in vacuum \( \omega_{\text{vac},1} \), and quality factor \( Q = Q_1 \), (14), for the fundamental mode. Upper row: \( \bar{T} = 0.005 \) (short-dashed line); \( \bar{T} = 0.015 \) (dashed line); \( \bar{T} = 0.005 \) (solid line). Lower row: \( \bar{T} = 45 \) (short-dashed line); \( \bar{T} = 15 \) (dashed line); \( \bar{T} = 5 \) (solid line). Results in the limit \( \overline{\text{Re}} \to \infty \) for \( \bar{T} = 0.045, 0.015, 0.005 \) are \( 1 - \omega_p/\omega_{\text{vac},1} = 0.0172, 0.00584, 0.00196 \), respectively. Results in the limit \( \overline{\text{Re}} \to \infty \) for \( \bar{T} = 45, 15, 5 \) are \( \omega_p/\omega_{\text{vac},1} = 0.166, 0.280, 0.451 \), respectively.
to note that a non-zero peak frequency is observed in all cases presented. Such behavior is not observed in a SHO model, where the peak frequency is found to be identically zero for all quality factors \( Q \leq 1/\sqrt{2} \). These results demonstrate that for \( Q \leq O(1) \) the frequency response of a cantilever beam is not analogous to that of a SHO.

2.2 Torsional oscillation

We now turn our attention to the torsional modes of oscillation, as presented by Green & Sader (2002). This analysis follows along analogous lines to that presented above for flexural oscillation. Due to this similarity, we only summarise the key results of this model. To begin, the vacuum frequencies of a cantilever beam executing torsional oscillations are given by

\[
\omega_{\text{vac},n} = \frac{D_n}{L} \sqrt{\frac{GK}{\rho_c I_p}},
\]

\[
D_n = \frac{\pi}{2} (2n - 1),
\]

where \( G \) is the shear modulus of the cantilever and \( n = 1, 2, 3, \ldots \). For a thin rectangular beam \( I_p = b^3 h/12 \) and \( K \) is the torsional constant of the cross-section, which in the case of a thin rectangle reduces to \( bh^3/3 \) (Roark, 1943).

The total moment per unit length \( M(x|\omega) \) applied to the beam is decomposed into a term corresponding to the driving moment \( M_{\text{drive}}(x|\omega) \) and the moment due to hydrodynamic loading of the cantilever \( M_{\text{hydro}}(x|\omega) \), i.e.,

\[
M(x|\omega) = M_{\text{hydro}}(x|\omega) + M_{\text{drive}}(x|\omega).
\]

The hydrodynamic load is then expressed generally as

\[
M_{\text{hydro}}(x|\omega) = -\frac{\pi}{8}\rho \omega^2 b^4 \Gamma^t(\omega) \Phi(x|\omega),
\]

where \( \Phi(x|\omega) \) is the deflection angle about the longitudinal axis of the cantilever and \( \Gamma^t(\omega) \) is the hydrodynamic function. The expression for the thermal noise spectrum of the cantilever is (van Eysden & Sader, 2008, 2009b)

\[
|\Phi(x|\omega)|^2_s = \frac{6\pi k_B T}{k_\Phi} \rho b \frac{D_1^2}{\rho_c h} \frac{\omega^2}{\omega_{\text{vac},1}^2} \sum_{n=1}^{\infty} \frac{\omega \Gamma^t_n(\omega)}{|D_2^2 - A_n^2(\omega)|^2} \gamma_n^2(x),
\]

where the subscript \( s \) again refers to the spectral density, \( k_\Phi \) is the torsional spring constant (Roark, 1943) and

\[
A_n(\omega) = \frac{\pi}{2} \frac{\omega}{\omega_{\text{vac},1}} \left( 1 + \frac{3\pi \rho b}{2\rho_c h} \Gamma^t(\omega) \right)^{1/4}.
\]

In the limit of small dissipative effects, the conditions for which are discussed above, the resonant frequency \( \omega_{R,n} \) and quality factor \( Q_n \) for the \( n^{\text{th}} \) mode of
vibration are given by

\[
\frac{\omega_{R,n}}{\omega_{vac,n}} = \left(1 + \frac{3\pi \rho_b}{2\rho_c h} \Gamma_t^i(\omega_{R,n}) \right)^{-1/2},
\]

(25)

\[
Q_n = \frac{2\rho_c h}{3\pi \rho_b} + \frac{\Gamma_t^i(\omega_{R,n})}{\Gamma_t^f(\omega_{R,n})}.
\]

(26)

The hydrodynamic function \(\Gamma^t(\omega)\) is obtained from the solution of the linearised Navier Stokes equation for flow around a rigid thin blade of infinite length that is executing torsional oscillation; this is analogous to the flexural oscillation case. The required result is given in Green & Sader (2002), and is derived using the boundary integral technique of Tuck (1969). An approximate numerical formula is provided for this result, which has the correct asymptotic behaviour as \(Re \to 0\) and \(Re \to \infty\), and is accurate to better than 0.1% over the range \(10^{-4} \leq Re \leq 10^{4}\). For details the reader is referred to Green & Sader (2002).

### 2.3 In-plane flexural oscillation

The model in §2.1 can be directly applied to cantilever beams of arbitrary cross section. All that is required is specification of the hydrodynamic function for the geometry under consideration. Brumley et al. (2010) calculated the hydrodynamic function for a rectangular cylinder of arbitrary width-to-thickness ratio. It was found that the hydrodynamic function is weakly affected by the beam thickness, for width-to-thickness ratios greater than unity – this shows that approximation of a beam of small, but finite thickness, by one that is infinitely thin is a good approximation for out-of-plane flexural oscillation; see §2.1. The situation is very different when the beam width is smaller than its thickness, corresponding to in-plane flexural oscillation. In this latter case, the width-to-thickness ratio can strongly affect the hydrodynamic load – this geometric ratio must be specified to yield accurate results, in general. To facilitate practical application, tabulated data for the hydrodynamic function, spanning a wide range of width-to-thickness ratios, is presented in Brumley et al. (2010). Cox et al. (2012) also examined the frequency response of the in-plane flexural modes of a cantilever beam of rectangular cross section.

### 2.4 Extensional oscillation

Theoretical models for the extensional modes of cantilever beams and free-free beams have also been derived (Pelton et al., 2009; Castille et al., 2010; Pelton et al., 2011; Chakraborty et al., 2013). These models build on the same principle developed for flexural oscillations, and approximate the hydrodynamic load at any position along the beam by that of a rigid cylinder. Since these modes typically operate at frequencies much higher than those generated by the fundamental flexural mode, the viscous boundary layer near the surface is much
smaller than the cantilever width. This, in turn, enables approximation of the hydrodynamic load by a local solution derived from Stokes’ 2nd problem for the oscillation of a half-space in a viscous fluid (Landau & Lifshitz, 1959). These models have been successfully compared to experimental measurements.

3  Arbitrary mode order

3.1  Incompressible flows

In the previous section, we examined the frequency response of cantilever beams of high aspect ratio (length/width) operating at low mode order, where the flow field is well approximated as being two-dimensional. As mode order increases, however, the spatial wavelength of the modes decreases which ultimately leads to violation of this two-dimensional approximation and breakdown in these models. Due to the importance of higher order modes in AFM and MEMS applications, a rigorous analysis of the three-dimensional flow field around a cantilever is vital. This is relevant to applications that utilize the higher order modes of cantilever beams (van Noort et al., 1999; Ulcinas & Sntka, 2001; Stark, 2004; Braun et al., 2005).

Finite-element fluid-structure models for cantilevers immersed in viscous fluids have been developed, e.g., see Paul & Cross (2004); Maali et al. (2005); Basak et al. (2006). Some of these models enable cantilever plates of arbitrary aspect ratio (length/width) to be investigated; however, all such models rely on sophisticated and computationally intensive numerical techniques. Since many cantilevers used in practice possess large aspect ratios, it is desirable to have theoretical models of an analytical nature that can be readily implemented, and this forms one aim of the present chapter. In addition to developing such models, we systematically investigate the effect of increasing mode number on the general characteristics of the frequency response, which is relevant to the design and operation of cantilevers in practice. Analytical solutions for the hydrodynamic functions and their use in development of analytical models for the frequency response of cantilevers in fluid, at arbitrary mode order, are reported in van Eysden & Sader (2006b, 2007); these models are summarized here.

We consider a beam of high aspect ratio (length/width) whose width greatly exceeds its thickness, as before. The hydrodynamic function is given by that of an infinitely long oscillating thin blade with deflection function $w(x, \omega) = Z_0 \exp(ikx)$; see van Eysden & Sader (2007). This yields the exact analytical solution for an incompressible viscous fluid (van Eysden & Sader, 2006b):

$$\Gamma_f(\omega, n) = 8a_1,$$

(27)

where the coefficient $a_1$ is obtained by solving the system of linear equations

$$\sum_{m=1}^{M} (A_{q,m}^e + A_{q,m}^{Re}) a_m = \begin{cases} 1 : & q = 1 \\ 0 : & q > 1 \end{cases},$$

(28)
for $1 \leq q \leq M$ and

$$A_{q,m}^e = -\frac{4^{2q-1}}{\sqrt{\pi}} G^{21}_{13} \left( \frac{\kappa^2}{16} \begin{array}{c} 0 \\ q + m - 1 \end{array} \begin{array}{c} 3 \frac{7}{2} \\ q - m + 1 \end{array} \right), \quad (29)$$

$$A_{q,m}^{Re} = -\kappa^2 \frac{2^{4q-5}}{\sqrt{\pi}} G^{21}_{13} \left( \frac{\kappa^2 - i \text{Re}}{16} \begin{array}{c} 0 \\ q + m - 2 \end{array} \begin{array}{c} 1 \frac{7}{2} \\ q - m \end{array} \right)$$

$$-\frac{2^{4q-1}}{\sqrt{\pi}} G^{21}_{13} \left( \frac{\kappa^2 - i \text{Re}}{16} \begin{array}{c} 0 \\ q + m - 1 \end{array} \begin{array}{c} 1 \frac{7}{2} \\ q - m + 1 \end{array} \right), \quad (30)$$

in terms of Meijer G-functions (Luke, 1969). The integer $M$ is to be increased until convergence in the solution is obtained; features of these solutions are discussed by van Eysden & Sader (2006b). The hydrodynamic function $\Gamma^f(\omega, n)$ depends on the radial frequency $\omega$ through the Reynolds number (9), and on the mode order $n$ through the dimensionless parameter

$$\kappa = C_n \frac{b}{L}, \quad (31)$$

where $C_n$ is as defined in (6). This parameter gives the relative aspect ratio of each spatial mode, and as such, is referred to as the “normalised mode number.” The asymptotic result for $\Gamma^f(\omega, n)$ in the limit as $\kappa \to 0$ is identical to (15). For the complementary limit where $\kappa \gg 1$, the hydrodynamic function has the asymptotic form,

$$\Gamma^f(\omega, n) = \frac{8}{\pi |\kappa|} \sqrt{\frac{\kappa^2 - i \text{Re}}{\kappa^2 - i \text{Re} - |\kappa|}}, \quad \kappa \to \infty. \quad (32)$$

In the inviscid limit, the matrix elements $A_{q,m} = A_{q,m}^e + A_{q,m}^{Re}$ are given by (van Eysden & Sader, 2009b)

$$A_{q,m} = -\frac{4^{2q-1}}{\sqrt{\pi}} G^{21}_{13} \left( \frac{\kappa^2}{16} \begin{array}{c} 0 \\ q + m - 1 \end{array} \begin{array}{c} 3 \frac{7}{2} \\ q - m \end{array} \right). \quad (33)$$

The dependence on $\omega$ and mode number, $n$, in (29) and (30) is contained in the argument of the Meijer G-function, and subsequently within the coefficients $a_n$. Similar formulas exist for the torsional modes of oscillation (van Eysden & Sader, 2009a).

Importantly, the hydrodynamic function now has an explicit dependence on the mode number $n$, and we make the replacements $\Gamma^f(\omega) \to \Gamma^f(\omega, n)$ and $\Gamma^f(\omega) \to \Gamma^f(\omega, n)$ in all equations in the previous section. Consequently, (8) now places no restriction on the mode order $n$ and rigorously accounts for the three-dimensional flow field around the beam (while ignoring end effects).

We first turn our attention to investigate the effect of increasing mode number on the frequency response of cantilever beams immersed in fluid. Importantly, the fundamental torsional resonant frequency greatly exceeds the fundamental flexural resonant frequency (Green & Sader, 2002), as is commonly
observed in measurements (Green et al., 2004). Since the higher order torsional modes are rarely probed in practice, we focus our discussion exclusively on the flexural modes. Nonetheless, we note that the influence of higher order mode number on the torsional frequency response will be very weak in comparison with the flexural modes, since the torsional hydrodynamic function $\Gamma_t(\omega, n)$ is much more weakly dependent on mode number, $n$, in comparison with the complementary dependence of the flexural hydrodynamic function $\Gamma_f(\omega, n)$ (van Eysden & Sader, 2006b). For further details regarding the torsional hydrodynamic function, the reader is referred to (van Eysden & Sader, 2006b).

To completely characterise the frequency response for higher order modes, the following dimensionless parameter is required in addition to $\bar{Re}$ and $\bar{T}$ defined in (18),

$$\bar{\kappa} = C_1 \frac{b}{L}. \quad (34)$$

This third parameter is a normalised aspect ratio and accounts for the induced three-dimensional hydrodynamic flow around the beam. In the singular limit of $\bar{\kappa} \to 0$, the low mode number theory of §2.1 is recovered.

In Figure 3, we present results for the thermal noise spectrum of the slope of the cantilever at its end point ($x = 1$), as this is the quantity of interest in AFM measurements. Results are presented in gas (left-hand side) and liquid (right-hand side) for $\bar{\kappa} = 0, 0.125, 0.25, 0.5$, which corresponds to a geometric aspect ratio of $L/b = \infty, 15, 7.5, 3.75$, respectively. In gases, we see that the frequency response in the neighbourhood of the fundamental resonance peak is very weakly dependent on the aspect ratio of the cantilever; see left-hand side of Figure 3. This is a result of the weak dependence of the hydrodynamic function on $\bar{\kappa}$ for the value of normalised Reynolds number considered; see Table 1. However, as the mode number increases, the thermal noise spectra begin to deviate significantly from the $\bar{\kappa} = 0$ solution, as seen in the lower left-hand panel of Figure 3. This can be understood in terms of the normalised mode number $\kappa$, which is increasing [see (31)], resulting in a significant change in the hydrodynamic function $\Gamma_f(\omega, n)$ [see Table 1 and (32)], which is reflected in the frequency response.

Comparing the left- and right-hand sides of Figure 3, we find that immersion in liquid has a dramatic effect in comparison with that of gas. The resonance peaks broaden and shift significantly to lower frequencies. Note that the peak frequency of the fundamental mode is now approximately four times lower than the resonant frequency in gas. Nonetheless, inspecting the upper right-hand panel of Figure 3 we find that the fundamental resonance peak is very weakly affected by changing the aspect ratio of the cantilever between $L/b = 3.75$ and $\infty$. The same is not true of the next mode, however, where a significant effect on both the peak frequency and quality factor is observed for the largest value of $\bar{\kappa} = 0.5$ (corresponding to the smallest value of $L/b = 3.75$). Interestingly, it is found that as the normalised aspect ratio $\bar{\kappa}$ is increased (resulting in a decrease in $L/b$), the normalised peak frequencies shift to higher frequencies and approach the vacuum frequencies. This effect increases with increasing mode number $n$. Importantly, even the smallest nonzero value of $\bar{\kappa}$ considered
Figure 3: Normalised thermal noise spectrum (slope) $H = |w'(x, \omega)|^2 k\omega_{vac,1}/(k_B T)$, (11), of the flexural modes in gas. The $'$ refers to the derivative with respect to $x$. Normalised mode numbers $\kappa = 0, 0.125, 0.25, \text{and} 0.5$ corresponding to $L/b = \infty, 15, 7.5, \text{and} 3.75, \text{respectively.}$ Solid line corresponds to $\kappa = 0$ result, and is identical to model in §2.1. Upper row: First mode (left), first and second modes (right); Lower row: First 6 modes. Results given for a gas with $Re = 10, T = 0.01$ (left-hand side) and liquid with $Re = 100, T = 10$ (right-hand side).
Table 1: Incompressible hydrodynamic function $\Gamma(f(w, \eta))$ for flexural modes as a function of Reynolds number $Re$ and normalised mode number $\kappa$, accurate to six significant figures. (a) Real component. Re $\to \infty$ results obtained using inviscid theory (33). (b) Imaginary component.

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(continued)
Figure 4: Normalised resonant frequency, (13), as a function of mode number $n$. Lines given as a guide. Low mode number model (solid circle), arbitrary mode number model (open circles), and large $\kappa$ asymptotic model (open rectangles). Upper row: $\bar{\kappa} = 0.0625$ corresponding to $L/b = 30$. Lower row: $\bar{\kappa} = 0.25$ corresponding to $L/b = 7.5$. Results given for gas $\text{Re} = 1$, $\bar{T} = 0.01$ (left-hand side) and liquid $\text{Re} = 10$, $\bar{T} = 10$ (right-hand side).

(corresponding to the largest finite value of $L/b = 15$) has a significant effect, as is strikingly evident in Figure 3 (lower right-hand panel). From these results it is clear that the frequency response of the higher modes of practical cantilevers can strongly depend on the normalised aspect ratio $\bar{\kappa}$. This finding for the higher modes contrasts directly to the fundamental mode that is found to be virtually independent of $\bar{\kappa}$.

The effect of varying aspect ratio on the resonant frequencies and quality factors is presented in Figures 4 and 5, respectively, where results for the first 20 modes are calculated using (i) the arbitrary mode number model, (ii) the low mode number model presented in §2.1, and (iii) the large $\kappa$ asymptotic model that is obtained using the hydrodynamic function in (32). For immersion in gas, we find that the resonant frequencies are very weakly affected by the surrounding fluid, in agreement with previous studies (Sader, 1998; Green & Sader, 2002; Paul & Cross, 2004; Maali et al., 2005; Basak et al., 2006; Paul et al., 2006; Chon et al., 2000); see Figure 4. We also observe that the arbitrary mode number model matches the low mode number model in the limit of $\bar{\kappa} = 0$ (corresponding to $L/b \to \infty$), but can diverge significantly as the aspect ratio, $L/b$, of the cantilever is reduced. Nonetheless, we find that the low mode number model is always valid for the fundamental mode for the range of normalised aspect ratios $\bar{\kappa}$ considered; this was also found to be the case up to $\bar{\kappa} = 0.5$, beyond which the underlying assumptions in all models must be drawn into question.
Figure 5: As for Figure 4, except results given for quality factor $Q_n$, (14), as a function of mode number $n$.

In Figure 4, the arbitrary mode number model is found to give identical results to the low mode number model for the fundamental mode then departs from this solution as the mode number $n$ is increased, ultimately approaching the large $\kappa$ asymptotic model. Similar results are also obtained for the resonant frequencies in liquid, although the deviation is much more pronounced due to the dramatic effect of liquid on the frequency response in comparison with gas, cf. the left- and right-hand sides of Figure 4.

Complementary results to Figure 4 are presented for the quality factor $Q_n$ in Figure 5. For all panels it is evident that as the mode number $n$ increases, the quality factor approaches infinity. This is a consequence of the diminishing hydrodynamic load at high mode number [see Table 1 and (32)], an effect which is also present in Figure 4, where the resonant frequencies in fluid approach the vacuum results. Again, we find that the low mode number model agrees well with the arbitrary mode number model for the fundamental mode always, but as the mode number $n$ is increased these results begin to depart from the low mode number model and approach the large $\kappa$ asymptotic model. It is interesting to note, however, that the deviation from the low mode number model for immersion in gas is much more pronounced than the deviation in liquid, cf. left- and right-hand panels of Figures 5. For all cases presented, it is found that the quality factor in liquid is well approximated by the low mode number model, regardless of the mode number; see (van Eysden & Sader, 2007).

Finally, we comment on the validity of the above model for practical can-
tilevers. In the $\kappa = 0$ limit, the low mode number model of §2.1 is valid for large $L/b$ provided the viscous boundary layer, defined as $b/\sqrt{\text{Re}}$, does not become comparable to the length of the beam, at which point the flow field is directly affected by the three-dimensional nature of the cantilever geometry. For the higher order modes, the relevant length scale is the spatial length scale of oscillations $L/C_n$, and the requirement that this is much less than the viscous boundary layer thickness can be expressed as

$$\text{Re} > \kappa^2.$$  \hspace{1cm} (35)

Approximating the resonant frequency of the cantilever by that in vacuum, we find that the condition (35) is independent of mode number and is equivalent to $\text{Re} > \kappa^2$. This condition is typically satisfied by practical microcantilevers, supporting the validity of the low mode number model of §2.1, for the fundamental mode. To examine the condition (35) carefully, we can evaluate the ratio

$$\frac{\text{Re}}{\kappa^2} = \frac{h}{\omega_{\text{vac}}/\omega_n} \sqrt{\frac{E}{12\rho_c}},$$  \hspace{1cm} (36)

which must be greater than unity. It is interesting that this ratio is independent of the cantilever length and width and only depends on its thickness. For cantilever materials and fluids typically used in AFM and MEMS applications (Binnig et al., 1986; Berger et al., 1997; Ho & Tai, 1998; Sader, 1998; Sader et al., 1999; Chon et al., 2000; Green & Sader, 2002; Paul & Cross, 2004; Lavrik et al., 2004; Maali et al., 2005; Basak et al., 2006; Paul et al., 2006), (36) can be estimated to give $\text{Re}/\kappa^2 \sim 10^8 h \omega_{R,n}/\omega_{\text{vac},n}$, where $h$ is the thickness in metres. This indicates that the above ratio greatly exceeds unity for microcantilevers of practical relevance. The ratio $\omega_{R,n}/\omega_{\text{vac},n}$ can be calculated using (13). Only in cases of very heavy fluid loading, or very thin cantilevers, where $\omega_{R,n}/\omega_{\text{vac},n} \ll 1$ will the ratio in (36) equal unity. Thus, for most cantilevers of practical relevance, the condition $\text{Re} \gg \kappa^2$ will be satisfied, indicating the arbitrary mode number model captures the true frequency response at resonance of these cantilever beams. Complete three-dimensional computational fluid dynamics simulations of the cantilever dynamics are not essential in such situations, even for very small cantilevers. Nonetheless, by tuning the material properties of the cantilever and fluid, it may be possible to reach this limit practically; see (36).

### 3.2 Compressible flows

In general, the flow around a cantilever can be considered incompressible provided the acoustic wavelength greatly exceeds the dominant length scale of the flow (Morse & Ingard, 1987; van Eysden & Sader, 2009a). For practical microcantilevers with high aspect ratios (length/width) and width-to-thickness ratios, operating at low mode numbers (i.e., the fundamental flexural mode and the next few modes), the flow is approximately two-dimensional and its dominant
length scale is the beam width, \( b \). In such cases, the acoustic wavelength greatly exceeds the dominant length scale, and hence, the incompressibility assumption is easily satisfied (Sader, 1998). This has been well established by the experimental validation of numerous theoretical models based on incompressible flow, e.g., see Chon et al. (2000); Basak et al. (2006); Ghatkesar et al. (2008). For operation at higher order modes, however, it is important to reassess how this incompressibility condition is affected.

To investigate the effects of fluid compressibility, the hydrodynamic function \( \Gamma f(\omega, n) \) must be generalised to include compressibility effects. A solution can be obtained using the model presented in the previous sections, where the incompressibility condition (assumption 5) above is relaxed. The exact analytical solution for the flow around an infinitely long oscillating flat blade in a compressible medium is presented in van Eysden & Sader (2009a). The result is obtained from (27) under the following replacement for (29):

\[
A_{q,m}^n = -\frac{4^{2q-1}}{\sqrt{\pi}} G_{13}^{21} \left[ \frac{1}{16} \left( \kappa^2 - \varsigma^2 \frac{Re}{Re - 4/3i\varsigma^2} \right) \right] 0 \quad q + m - 1 \quad q - m + 1, \quad (37)
\]

while (30) is unchanged. In the inviscid limit, (33) generalises to

\[
A_{q,m} = -\frac{4^{2q-1}}{\sqrt{\pi}} G_{13}^{21} \left( \frac{\kappa^2 - \varsigma^2}{16} \right) 0 \quad q + m - 1 \quad q - m \quad (38)
\]

The hydrodynamic function \( \Gamma f(\omega, n) \) is dimensionless and depends on fluid compressibility through the dimensionless parameter

\[
\varsigma = \frac{\omega b}{c}, \quad (39)
\]

where \( c \) is the sound speed of the fluid. Therefore, \( \varsigma \) is a normalised wave number for the propagation of sound in the medium and is a measure of the fluid compressibility. Note that for the continuum approximation to be valid, we require \( \varsigma \ll Re \) (Landau & Lifshitz, 1959) and the argument of the Meijer G-function in the first term of (37) reduces to \((\kappa^2 - \varsigma^2)/16\) for cases of practical interest; see van Eysden & Sader (2009a). In the limit as \( \varsigma \to 0 \), the incompressible result for \( \Gamma f(\omega, n) \) presented above is obtained.

### 3.2.1 Scaling analysis

The conditions for which fluid compressibility becomes important can be determined using a scaling analysis. For an elastic beam, the dominant length scale of the flow is the minimum of (i) the beam width \( b \) and (ii) the length scale of spatial oscillations of the beam, given by

\[
\lambda_{beam} = \frac{2\pi L}{C_n} \approx \frac{4L}{2n - 1}, \quad (40)
\]

Since \( C_n \) increases with increasing mode number \( n \), the length scale of spatial oscillations reduces until it becomes smaller than the beam width. Thus, spatial
oscillations of the beam will eventually set the dominant hydrodynamic length scale, as the mode number is increased.

The wavelength of acoustic oscillations generated by the beam in a fluid is $c/f_n$, where $f_n$ is the resonant frequency. For the purpose of a scaling analysis, the resonant frequency can be approximated by the corresponding result in vacuum, i.e., (5). The wavelength of sound is then

$$\lambda_{\text{sound}} = \left( \frac{C_1}{C_n} \right)^2 \frac{c}{f_{\text{vac},1}} \approx \frac{1.425}{(2n-1)^2} \frac{c}{f_{\text{vac},1}} ,$$

(41)

where $\omega_{\text{vac},1} = 2\pi f_{\text{vac},1}$. Therefore, the acoustic wavelength decreases in inverse proportion to the square of the mode number $n$, for high mode numbers. Critically, this has a stronger dependence on mode number than the length scale of spatial oscillations. Therefore, as the mode number increases, the acoustic wavelength eventually becomes comparable to the spatial wavelength of the beam, and compressibility cannot be ignored. This so-called coincidence point corresponds to the onset of significant radiation damping of the beam, where energy is carried away in the form of sound waves (Morse & Ingard, 1987; van Eysden & Sader, 2009a).

Equating the expressions in (40) and (41), determines the critical mode number $n_c$ at which coincidence is reached,

$$n_c = \frac{0.178c}{f_{\text{vac},1} L} .$$

(42)

For a cantilever 250 $\mu$m in length with a vacuum frequency of 20 kHz, immersed in air, coincidence is reached at $n_c \approx 12$. In liquids, however, the speed of sound is a factor of $\sim 5$ larger than in air and the critical mode number is approximately $n_c \approx 60$, which is unlikely to be probed in practice. The precise regime for which the effects of fluid compressibility become manifest is clarified below using the numerical results of the above model.

### 3.2.2 Numerical results

To completely characterise the frequency response for higher order modes, the following dimensionless parameter is required in addition to $\bar{\Re}$, $\bar{T}$ and $\bar{\kappa}$ defined in (18) and (34),

$$\bar{\varsigma} = \frac{b \omega_{\text{vac},1}}{c} .$$

(43)

The parameter $\bar{\varsigma}$ is the normalised wave number for the propagation of sound through the medium, and is a measure of the fluid compressibility. When the acoustic wavelength is large, this parameter approaches zero and the incompressible limit is attained.

Another dimensionless parameter of fundamental importance in compressible flow is $\bar{\varsigma}/\bar{\kappa}$, which is the ratio of the beam length to the acoustic wavelength, for the fundamental mode. Since (42) can be written as

$$n_c = \frac{0.596\bar{\kappa}}{\bar{\varsigma}} ,$$

(44)
Figure 6: Normalised thermal noise spectrum (slope) $H = |w' (x, \omega)|^2 \frac{\omega \nu_{\text{vac}, 1}}{(k_B T)}$ for the flexural modes, showing the effects of fluid compressibility. The $'$ refers to the derivative with respect to $x$. Results given for (left-hand panel) gas: $Re = 5$, $T = 0.01$, $\kappa = 0.125$ and $\zeta/\kappa = 0, 0.025, 0.05, 0.075, 0.1$; (right-hand panel) liquid: $Re = 10$, $T = 10$, $\kappa = 0.125$ and $\zeta/\kappa = 0, 0.005, 0.01, 0.015, 0.02$.

it then follows that $\zeta/\kappa$ determines the mode number at which coincidence is reached. The dimensionless parameter $\zeta/\kappa$ will be used in the following results to vary the effects of compressibility.

In Figure 6, we present the thermal noise spectrum for the slope (angle) at the end of the cantilever for compressible flow. Results are given for practical values of $\kappa$, $\zeta$, and $Re$ in gases [Figure 6, left-hand panel] and liquids [Figure 6, right-hand panel], which clearly demonstrate the effect of increasing compressibility. Figure 6 (left-hand panel) shows that increasing the gas compressibility broadens the resonance peaks, which corresponds to enhanced dissipation. This effect is due to the generation of sound waves that radiate from the cantilever, which manifests itself more strongly with increasing mode number. This enhancement in dissipation is in direct contrast to the incompressible limit, where the effects of increasing mode number reduce dissipation (see Figure 4); this latter effect is due purely to the increasing importance of three-dimensional flow. Compressibility therefore counteracts the effects of three-dimensional flow with increasing mode number. This feature is evident in (37), where the variables $\kappa$ and $\zeta$ appear in combination.

From the right-hand panel of Figure 6, we observe that compressibility has virtually no effect on the thermal noise spectrum for cantilevers operating in a liquid, as the spectra are indistinguishable from the incompressible result (all five curves coincide). This supports the above scaling argument that compressibility does not become significant in liquids until very high mode numbers are attained; for typical microcantilevers, this corresponds to mode numbers in excess of 20. We therefore focus the following discussion on the behaviour of cantilevers in gas.

In Figure 7, the resonant frequencies and quality factors as a function of mode number are presented. On the left-hand side, values of $\kappa$ and $\zeta$ correspond approximately to those used in the above scaling analysis. On the right-hand side a larger aspect ratio of $L/b = 30$, corresponding to $\kappa = 0.0625$, is presented.
Figure 7: Upper row: Normalised resonant frequency $1 - \omega_{R,n}/\omega_{vac,n}$. Lower row: Quality factor $Q_n$ as a function of mode number $n$. Results given for a gas with $Re = 5$, $T = 0.01$, $\zeta/\kappa = 0.05$ and $\kappa = 0.125$ (left-hand side) and $\kappa = 0.0625$ (right-hand side). Lines are given as a guide only. Complete solution (solid circles), viscous incompressible solution (open circles), and inviscid compressible solution (open rectangles).
Three solutions are given: (i) the \textit{viscous incompressible} solution (open circles) of §3.1, (ii) the \textit{inviscid compressible} solution (open rectangles), (38), and (iii) the \textit{complete (viscous compressible)} solution (closed circles), (37). For low mode numbers, compressibility effects are weak and the viscous incompressible and complete solutions coincide. However, as the mode number increases, the viscous incompressible solution begins to deviate from the complete solution. On the left-hand side, coincidence is reached at mode number 14, where the inertial component of the hydrodynamic load reaches a maximum, in agreement with predictions from the scaling analysis. However, on the right-hand side, the coincidence point now occurs at \( n \sim 17 \), despite (44) predicting the same result; the discrepancy arising from the approximate nature of the scaling analysis, which neglects the fluid loading on the beam. Above coincidence, sound waves are strongly generated in the medium, and a further increase in mode number results in rapid diminishing of this (inertial) hydrodynamic load. For large mode numbers, the complete solution coincides with inviscid compressible result, as the viscous penetration depth over which vorticity diffuses decreases with increasing mode number.

Corresponding results for the quality factor in gas are given in the lower row of Figure 7. As for the frequency response, the quality factor is dominated by incompressible viscous effects at low mode numbers, and approaches the inviscid compressible solution for large mode numbers, with the transition occurring approximately at coincidence. Importantly, above coincidence we find that the quality factor reduces significantly with increasing mode number. This finding differs greatly from the incompressible result and is due to the strong generation of acoustic waves that greatly enhance dissipation, i.e., radiation damping. This non-monotonic behaviour of the quality factor with increasing mode number should be observable experimentally.

The above theoretical models for compressible flow have been derived using the hydrodynamic function for an infinitely long beam executing uniform oscillations with discrete spatial wave numbers. We conclude by commenting on the practical implications of these models to cantilever beams of finite length. In the limit where the acoustic wavelength is much greater than the length of the cantilever beam, the flow is incompressible and the validity of these models is well established (Elmer & Dreier, 1997; Chon et al., 2000; Green & Sader, 2002; Paul & Cross, 2004; Maali et al., 2005; Braun et al., 2005; Basak et al., 2006; Paul et al., 2006; van Eysden & Sader, 2006a, 2007; Ghatkesar et al., 2008); this is the practical case for operation in the low order modes. In the opposite situation where the acoustic wavelength is smaller than the spatial wavelength of beam oscillations, i.e., above coincidence, the effects of compressibility are localised and hence unaffected by the finite length of the beam; this corresponds to the high mode number limit. Therefore, the cantilever beam model is strictly valid in the two limits of small and large acoustic wavelengths relative to the dominant hydrodynamic length scale. In the intermediate regime where the acoustic wavelength is larger than, but comparable to the spatial wavelength of beam oscillations, the flow probes a spectrum of spatial wave numbers due to finite beam length; this has not been rigorously considered here. Nonetheless,
since the above solution holds rigorously in the bounding asymptotic limits of small and large wavelengths, it is expected to yield a good approximation in the intermediate regime.

References


Green, C. P. & Sader, J. E., 2002, Torsional frequency response of cantilever beams immersed in viscous fluids with applications to the atomic force microscope, *J. Appl. Phys.* 92, 6262

Green, C. P. & Sader, J. E., 2005, Frequency response of cantilever beams immersed in viscous fluids near a solid surface with applications to the atomic force microscope, *J. Appl. Phys.* 98, 114913


Sader, J. E., 1998, Frequency response of cantilever beams immersed in viscous fluids with applications to the atomic force microscope, *J. Appl. Phys.* 84, 64


Ulcinas, A. & Snitka, A., 2001, Intermittent contact AFM using the higher modes of weak cantilever, *Ultramicroscopy* 86, 217


van Eysden, C. A. & Sader, J. E., 2009a, Compressible viscous flows generated by oscillating flexible cylinders, Phys. Fluids 21, 013104
