Gravitational waves from hydrodynamic instabilities

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The compressible Navier-Stokes equation with an isothermal equation of state can be written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\left(h + \tfrac{1}{2}\boldsymbol{u}^2\right) - \boldsymbol{\omega}\times\boldsymbol{u} + \boldsymbol{f} + \boldsymbol{F}_{\rm visc}, \quad (1)$$

$$\frac{\partial h}{\partial t} = -\boldsymbol{u}\cdot\boldsymbol{\nabla} h - c_{\rm s}^2 \boldsymbol{\nabla}\cdot\boldsymbol{u}, \quad (2)$$

where

$$\boldsymbol{F}_{\rm visc} = \nu\left(\nabla\boldsymbol{u}^2 + \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u} + 2\mathsf{S}\cdot\boldsymbol{\nabla}\ln\rho\right) \quad (3)$$

is the viscous force.

1 Stationary forcing

We adopt a generalized ABC-flow forcing

$$\boldsymbol{f} = \frac{f_0}{\mathcal{N}}\begin{pmatrix} C\sin kz + \sigma B\cos ky \\ A\sin kx + \sigma C\cos kz \\ B\sin ky + \sigma A\cos kx \end{pmatrix} \quad (4)$$

where $\mathcal{N}^2 = (A^2+B^2+C^2)(1+\sigma^2)/2$ is a normalization constant and $f_0 = \langle \boldsymbol{f}^2\rangle^{1/2}$ is the rms value of the forcing. We define the Reynolds number as

$$\mathrm{Re} = u_{\rm rms}/\nu k_{\rm f} \quad (5)$$

where $k_{\rm f} = \sqrt{3}k$ is the effective forcing wavenumber. For $\sigma = 1$, we have the standard ABC flow, but we also allow for other values with $-1 \le \sigma \le 1$. The flow has positive (negative) helicity for $\sigma > 0$ (< 0) and is fully helical for $\sigma = \pm 1$. For $\sigma = 0$, we have the Archontis flow with zero helicity. In particular, for $A = 0$ and $B = C = 0$, we have a Beltrami flow when $\sigma = \pm 1$ and the Kolmogorov flow for $\sigma = 0$.

In the laminar phase, at small values of Re, the flow is fully helical, i.e., the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is parallel to $\boldsymbol{u}$ with $\boldsymbol{\omega} = k\boldsymbol{u}$. Therefore, $\boldsymbol{\omega}\times\boldsymbol{u} = \boldsymbol{0}$, so the only nonlinearity comes from the dynamical pressure term, $\boldsymbol{u}^2/2$. However, for the ABC flow forcing, $\boldsymbol{u}^2 = \mathrm{const}$. Therefore, saturation occurs only when $u_{\rm rms} = f_0/\nu k_{\rm f}^2$. The temporal evolution is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{\rm rms} = f_0 - \nu k_{\rm f}^2 u_{\rm rms} \quad \text{(laminar)}. \quad (6)$$

The solution is given by

$$u_{\rm rms}(t) = f_0\left(1 - e^{-\nu k_{\rm f}^2 t}\right). \quad (7)$$

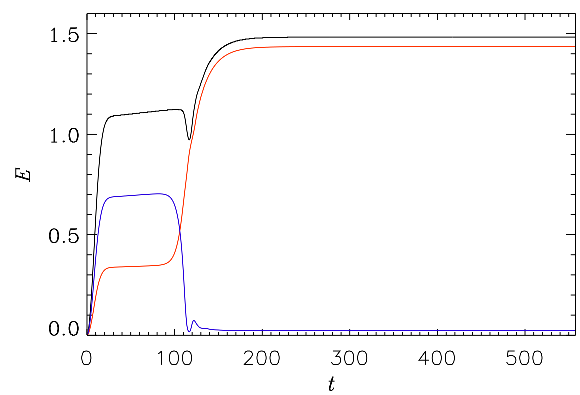

Figure 1: Total kinetic energy (black), kinetic energy in the xy-averaged velocity (red), and kinetic energy in the z-averaged velocity (blue). (`paver`)

Table 1:

Run	ν	f_0	Re
A	5×10^{-2}	0.1	25
B	10^{-3}	2×10^{-3}	100
B	10^{-3}	5×10^{-4}	100
C	5×10^{-4}	5×10^{-4}	25
C	5×10^{-4}	2×10^{-4}	25
C	2×10^{-4}	2×10^{-4}	25
C	5×10^{-4}	5×10^{-4}	25

Table 2:

Run	ν	f_0	Re	t_1	$\mathcal{E}_{\rm K}^{\max}$	$\mathcal{E}_{\rm GW}^{\max}$	$(\boldsymbol{\omega}\times\boldsymbol{u})_{\rm rms}^{\max}$	$(\boldsymbol{\nabla}\cdot\boldsymbol{u})_{\max}^2$
E	2×10^{-4}	2×10^{-4}	–	1121	0.017	9.0×10^{-13}	0.109	6.0×10^{-5}
B	1×10^{-3}	5×10^{-4}	–	894	0.037	2.1×10^{-11}	0.177	7.0×10^{-4}
H	5×10^{-4}	9×10^{-4}	–	553	0.066	1.3×10^{-9}	0.372	7.8×10^{-3}
I	5×10^{-4}	8×10^{-4}	–	581	0.059	5.9×10^{-10}	0.318	3.7×10^{-3}
F	5×10^{-4}	7×10^{-4}	–	638	0.054	6.0×10^{-10}	0.259	1.9×10^{-3}
C	5×10^{-4}	5×10^{-4}	–	783	0.041	9.9×10^{-11}	0.215	7.6×10^{-4}
D	5×10^{-4}	2×10^{-4}	–	1348	0.017	5.0×10^{-13}	0.070	4.4×10^{-5}
K	5×10^{-4}	1.4×10^{-4}	–	1653	0.0112	1.3×10^{-13}	0.047	1.1×10^{-5}
G	5×10^{-4}	1×10^{-4}	–	2098	0.0076	2.1×10^{-14}	0.026	3.2×10^{-6}

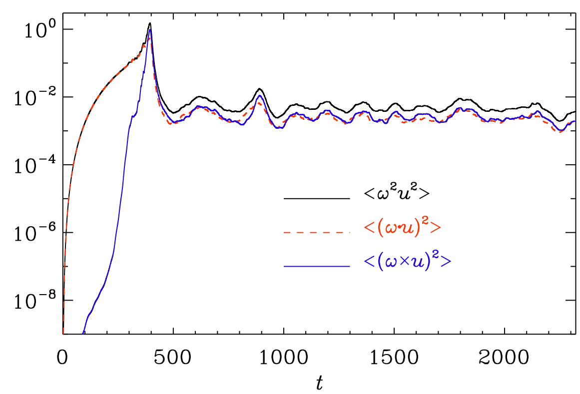

Figure 2: Evolution of $\langle\boldsymbol{\omega}^2\boldsymbol{u}^2\rangle$ (black) and it contributions $\langle(\boldsymbol{\omega}\cdot\boldsymbol{u})^2\rangle$ (red) and $\langle(\boldsymbol{\omega}\times\boldsymbol{u})^2\rangle$ (blue). (`phel_256e`)

References

Podvigina, O., & Pouquet, A. 1994, Phys. D, 75, 471

A Green's function

Monochromatic

$$\ddot{h}+k^2h=T \tag{8}$$

Solution

$$h(t)=k^{-1}\int_0^t \sin k(t-t')\,T(t')\,{\rm d}t' \tag{9}$$

The first derivative is

$$\dot{h}(t)=\int_0^t \cos k(t-t')\,T(t')\,{\rm d}t' \tag{10}$$

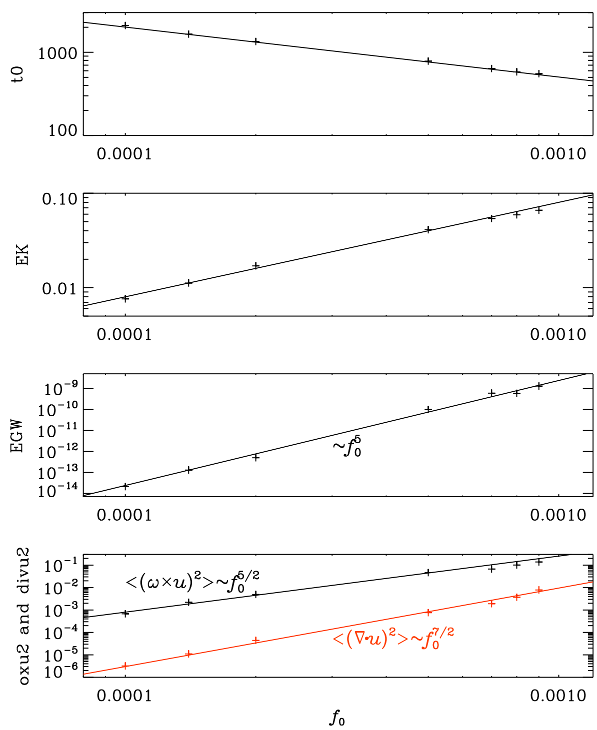

Figure 3: Results

where the derivative on the integral boundary gives zero, because the resulting term is proportional to Second derivative

$$\ddot{h}(t)=T(t)-k\int_0^t \sin k(t-t')\,T(t')\,{\rm d}t' \tag{11}$$

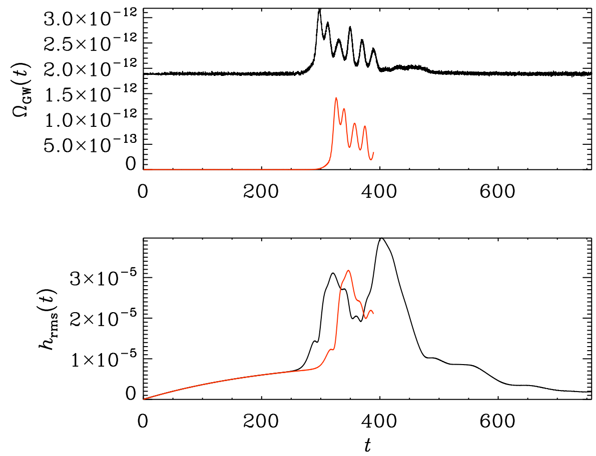

Figure 5: Dependence on noise for the Kolmogorov flow.

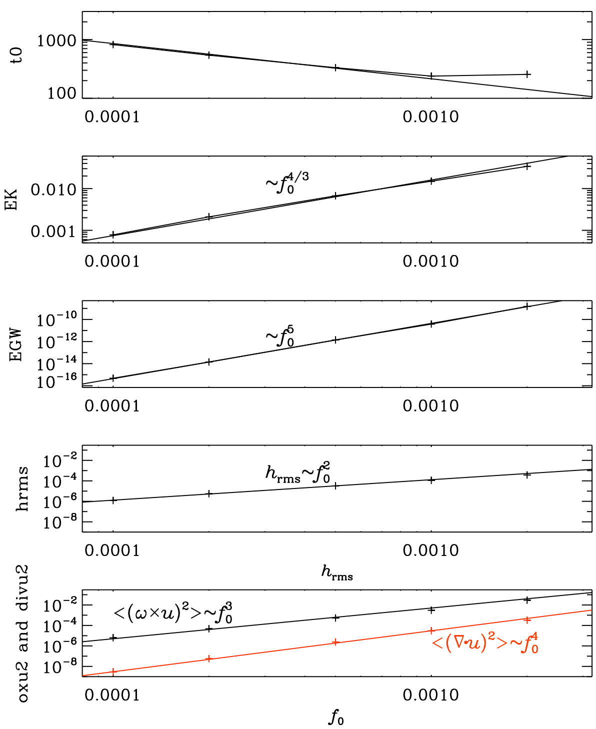

Figure 6: Results for the Kolmogov flow.

Let us now assume that $T(t)$ is given by $\boldsymbol{u}^2$ using Eq. (7), so that

$$h(t) = \frac{f_0^2}{k} \int_0^t \sin k(t-t') \left(1 - e^{-2\nu k_{\rm f}^2 t'}\right) \mathrm{d}t' \quad (12)$$

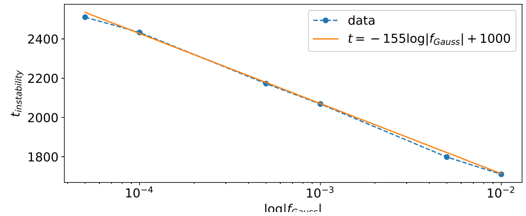

Figure 7: Time dependence of instability on Gaussian noise (kinetic drive)

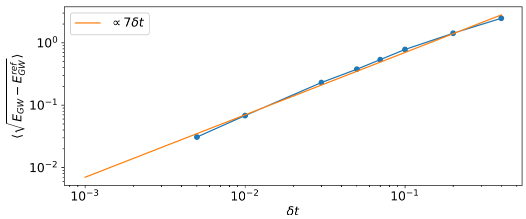

Figure 8: Error dependence on time step (kinetic drive)

$$\dot{h}(t) = f_0^2 \int_0^t \cos k(t-t') \left(1 - e^{-2\nu k_{\rm f}^2 t'}\right) \mathrm{d}t' \quad (13)$$

Using Ptolemy's identities,

$$\sin k(t-t') = \sin kt \cos kt' - \cos kt \sin kt', \quad (14)$$

$$\cos k(t-t') = \cos kt \cos kt' - \sin kt \sin kt', \quad (15)$$

and the integrals

$$\int_0^t \cos kt' e^{-2\nu k_{\rm f}^2 t'} \,\mathrm{d}t' = \frac{e^{-2\nu k_{\rm f}^2 t}}{\sqrt{\omega^2 + 4\nu^2 k_{\rm f}^4}} \cos(\omega t - \phi) \quad (16)$$

$$\int_0^t \sin kt' e^{-2\nu k_{\rm f}^2 t'} \,\mathrm{d}t' = \frac{e^{-2\nu k_{\rm f}^2 t}}{\sqrt{\omega^2 + 4\nu^2 k_{\rm f}^4}} \sin(\omega t - \phi) \quad (17)$$

where $\phi = -1/(1 + \omega^2/4\nu^2 k_{\rm f}^4)^{1/2}$. Using again Ptolemy's identities, we find

$$h(t) = \frac{f_0^2}{k} \left(1 - \frac{e^{-2\nu k_{\rm f}^2 t}}{\sqrt{\omega^2 + 4\nu^2 k_{\rm f}^4}} \sin\phi\right), \quad (18)$$

$$\dot{h}(t) = -\frac{f_0^2}{k} \frac{e^{-2\nu k_{\rm f}^2 t}}{\sqrt{\omega^2 + 4\nu^2 k_{\rm f}^4}} \cos\phi. \quad (19)$$

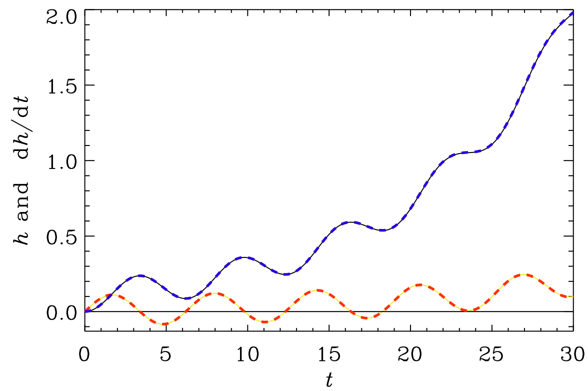

Figure 9: Evolution of h (solid black and dashed blue) and $\dot{h}$ (solid yellow and dashed red) for $\gamma = 0.1$, $T_0 = 0.1$, and $k = 1$. The solid lines indicate the numerical solution and the dashed lines indicate the analytic solution.

B Exponential growth

Let us now assume that $T(t)$ is given by $T_0(k)\, e^{\gamma t}$, so that

$$h(t) = \frac{T_0(k)}{k} \int_0^t \sin k(t-t')\, e^{\gamma t'}\, \mathrm{d}t' \qquad (20)$$

Using $\sin\phi = (e^{\mathrm{i}\phi} - e^{-\mathrm{i}\phi})/2\mathrm{i}$, we have

$$h(t) = \frac{T_0(k)}{2\mathrm{i}k} \int_0^t \left[e^{(\gamma-\mathrm{i}k)t'+\mathrm{i}kt} - e^{(\gamma+\mathrm{i}k)t'-\mathrm{i}kt} \right] \mathrm{d}t' \qquad (21)$$

Integrating between the two boundaries yields

$$h(t) = \frac{T_0(k)}{2\mathrm{i}k} \left[\frac{e^{\gamma t} - e^{\mathrm{i}kt}}{\gamma - \mathrm{i}k} - \frac{e^{\gamma t} - e^{-\mathrm{i}kt}}{\gamma + \mathrm{i}k} \right] \qquad (22)$$

Expanding this yields

$$h(t) = \frac{T_0(k)}{2\mathrm{i}k(\gamma^2+k^2)} \Big[(\gamma+\mathrm{i}k)(e^{\gamma t} - e^{\mathrm{i}kt}) - (\gamma-\mathrm{i}k)(e^{\gamma t} - e^{-\mathrm{i}kt}) \Big] \qquad (23)$$

or

$$h(t) = \frac{T_0(k)}{2\mathrm{i}k(\gamma^2+k^2)} \Big[\gamma(e^{\gamma t} - e^{\mathrm{i}kt}) + \mathrm{i}k(e^{\gamma t} - e^{\mathrm{i}kt}) - \gamma(e^{\gamma t} - e^{-\mathrm{i}kt}) + \mathrm{i}k(e^{\gamma t} - e^{-\mathrm{i}kt}) \Big] \qquad (24)$$

or

$$h(t) = \frac{T_0(k)}{2\mathrm{i}k(\gamma^2+k^2)} \Big[-\gamma(e^{\mathrm{i}kt} - e^{-\mathrm{i}kt}) + 2\mathrm{i}ke^{\gamma t} - \mathrm{i}k(e^{\mathrm{i}kt} + e^{-\mathrm{i}kt}) \Big] \qquad (25)$$

or

$$h(t) = \frac{T_0(k)}{k(\gamma^2+k^2)} \left(-\gamma \sin kt + ke^{\gamma t} - k\cos kt \right) \qquad (26)$$

and therefore

$$h(t) = \frac{T_0(k)}{\gamma^2+k^2} \left(e^{\gamma t} - \cos kt - \frac{\gamma}{k} \sin kt \right) \qquad (27)$$

with the derivative being

$$\dot{h}(t) = \frac{T_0(k)}{\gamma^2+k^2} \left[\gamma \left(e^{\gamma t} - \cos kt \right) + k \sin kt \right] \qquad (28)$$

Delta correlated $T(k) \sim \delta(k)$ leads to $\mathrm{Sp}(T) \sim k^2$, and $\delta(k)/k$ leads to $\mathrm{Sp}(T) \sim k^{-2}$. In simulations with chiral magnetic effect, we have $\gamma = \eta\mu_0^2$, so the critical k is $k \sim \eta\mu_0^2/c$.

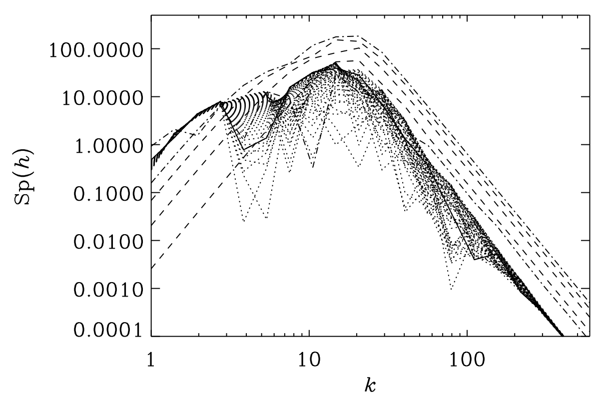

Figure 10: Solid line: last time, dotted lines: last 20 times, dashed lines: early times during growth phase, dashed-dotted lines: early decay phase.

C Numerical solution

Figure 10 shows a numerical solution for an initial spectrum of the source $\propto k^4/[1+(k/k_\mathrm{f})^6]$, where $k_\mathrm{f} = 20$ was chosen. The time evolution of the source was $T = t$ for $0 < t < 1$ and $1/[1+(t-1)/\tau]^2$ for $t > 1$.

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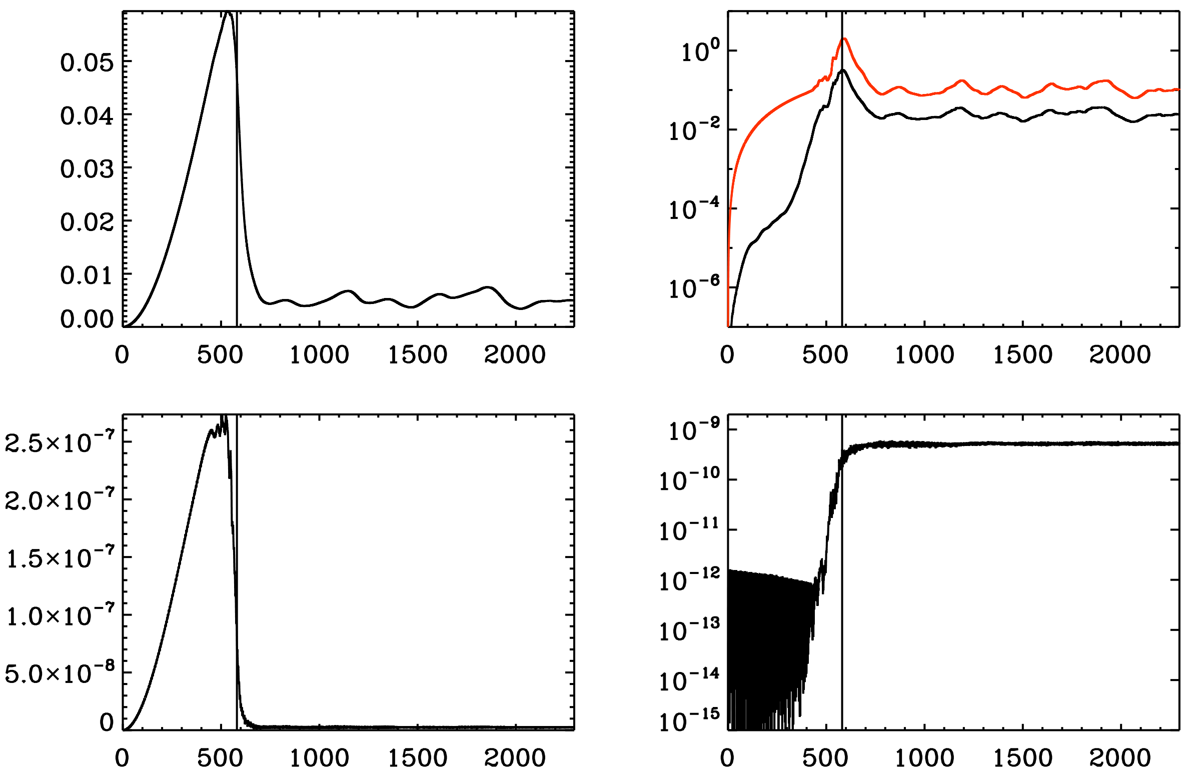

Figure 4: Results