

Mean-field transport in stratified and/or rotating turbulence

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ABSTRACT

Context. The large-scale magnetic fields of stars and galaxies are often described in the framework of mean-field dynamo theory. At moderate magnetic Reynolds numbers, the transport coefficients defining the mean electromotive force can be determined from simulations. This applies analogously also to the mean-field theory of passive scalar transport.

Aims. The mean electromotive force is investigated in the kinematic framework, that is, ignoring the back-reaction of the magnetic field on the fluid velocity, under the assumption of axisymmetric turbulence determined by the presence of either rotation, density stratification, or both (providing the corresponding directions are aligned to each other). As for the mean passive scalar flux, an analogous approach is used. As an alternative to convection, forced turbulence in an isothermal layer is considered. When using standard ansatzes, the mean magnetic transport is then determined by nine, and the mean passive scalar transport by four coefficients. We give results for all these transport coefficients.

Methods. The test-field method and the test-scalar method are used, where transport coefficients are determined by solving sets of equations with properly chosen mean magnetic fields or mean scalars. These methods are adapted to mean fields which may depend on all three space coordinates.

Results. Anisotropy of turbulent diffusion is found to be moderate in spite of rapid rotation or strong density stratification. Contributions to the mean electromotive force determined by the symmetric part of the gradient tensor of the mean magnetic field, which were ignored in several earlier investigations, turn out to be important. In stratified rotating turbulence, the α effect is strongly anisotropic, suppressed along the rotation axis on large length scales, but strongly enhanced at intermediate length scales. Also the $\Omega \times \mathbf{J}$ effect is enhanced at intermediate length scales. The turbulent passive scalar diffusivity is typically almost twice as large as the magnetic turbulent diffusivity. Both magnetic and passive scalar diffusion are slightly enhanced along the rotation axis, but decreased if there is gravity.

Conclusions. The test-field and test-scalar methods provide powerful tools for analyzing transport properties of axisymmetric turbulence. Future applications are proposed ranging from anisotropic turbulence due to the presence of a uniform magnetic field to inhomogeneous turbulence where the specific entropy is nonuniform, for example. Some of the contributions to the mean electromotive force which have been ignored in several earlier investigations, in particular those given by the symmetric part of the gradient tensor of the mean magnetic field, turn out to be of significant magnitude.

Key words. magnetohydrodynamics (MHD) – hydrodynamics – turbulence – Sun: dynamo

1. Introduction

Stellar mixing length theory is a rudimentary description of turbulent convective energy transport. The mixing length theory of turbulent transport goes back to Prandtl (1925) and, in the stellar context, to Vitense (1953). The simplest form of turbulent transport is turbulent diffusion, which quantifies the mean flux of a given quantity, e.g., momentum, concentration of chemicals, specific entropy or magnetic fields, down the gradient of its mean value. In all these cases essentially a Fickian diffusion law is established, where the turbulent diffusion coefficient is proportional to the rms velocity of the turbulent eddies and the effective mean free path of the eddies or their correlation length.

Mean-field theories, which have been elaborated, e.g., for the behavior of magnetic fields or of passive scalars in turbulent media, go beyond this concept. In the case of magnetic fields, the effects of turbulence occur in a mean electromotive force, which is related to the mean magnetic field and its derivatives in a tensorial fashion. Examples for

effects described by the mean magnetic field alone, without spatial derivatives, are the α -effect (Steenbeck et al., 1966) and the pumping of mean magnetic flux (Rädler, 1966, 1968; Roberts & Soward, 1975); for more information on these topics see, e.g., Krause & Rädler (1980) or Brandenburg & Subramanian (2005). Likewise the mean passive scalar flux contains a pumping effect (Elperin et al., 1996). In both the magnetic and the passive scalar cases turbulent diffusion occurs, which is in general anisotropic. The coupling between the mean electromotive force and the magnetic field and its derivatives, or mean passive scalar flux and the mean scalar and its derivatives, is given by turbulent transport coefficients.

On the analytic level of the theory the determination of these transport coefficients is only possible with some approximations. The most often used one is the second-order correlation approximation (SOCA), which has delivered so far many important results. Its applicability is however restricted to certain ranges of parameters like the magnetic

Reynolds number or the Péclet number. In spite of this restriction, SOCA is an invaluable tool, because it allows a rigorous treatment within the limits of its applicability. It is in particular important for testing numerical methods that apply in a wider range.

In recent years it has become possible to compute the full set of turbulent transport coefficients numerically from simulations of turbulent flows. The most accurate method for that is the test-field method (Schrinner et al., 2005, 2007). In addition to the equations describing laminar and turbulent flows, one solves a set of evolution equations for the small-scale magnetic or scalar fields which result from given mean fields, the test fields. By selecting a sufficient number of independent test fields, one obtains a corresponding number of mean electromotive forces or mean scalar fluxes and can then compute in a unique way all the associated transport coefficients.

Most of the applications of the test-field method are based on spatial averages that are taken over two coordinates. In the magnetic case this approach has been applied to a range of different flows including isotropic homogeneous turbulence (Sur et al., 2008; Brandenburg et al., 2008a), homogeneous shear flow turbulence (Brandenburg et al., 2008b) without and with helicity (Mitra et al., 2009), and turbulent convection (Käpylä et al., 2009). One of the main results is that in the isotropic case, for magnetic Reynolds numbers R_m larger than unity, the turbulent diffusivity is given by $\frac{1}{3}\tau u_{\text{rms}}^2$, where the correlation time τ is, to a good approximation, given by $\tau = (u_{\text{rms}} k_f)^{-1}$. Here, u_{rms} is the rms velocity of the turbulent small-scale flow and k_f is the wavenumber of the energy-carrying eddies. For smaller R_m , the turbulent diffusivity grows linearly with R_m . Furthermore, if the turbulence is driven isotropically by polarized waves, the flow becomes helical and there is an α effect. In the kinematic regime (for weak magnetic fields), the α coefficient is proportional to $\overline{\boldsymbol{\omega} \cdot \mathbf{u}}$, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity of the small-scale flow, \mathbf{u} . In the passive scalar case, test scalars are used to determine the transport coefficients. Results have been obtained for anisotropic flows in the presence of rotation or strong magnetic fields (Brandenburg et al., 2009), linear shear (Madarassy & Brandenburg, 2010), and for irrotational flows (Rädler et al., 2011).

The present paper deals with the magnetic and the passive scalar case in the above sense. Its goal is to compute the transport coefficients for axisymmetric turbulence, that is, turbulence with one preferred direction, given by the presence of either rotation or density stratification or, if the relevant directions coincide, of both. (Axisymmetric turbulence can be defined by requiring that any averaged quantity depending on the turbulent velocity field is invariant under any rotation of this field about the preferred axis.) Note that a dynamo-generated magnetic field will in general violate the assumption of axisymmetric turbulence. To avoid this problem while still being able to investigate the general effects arising from only one preferred direction, we assume such fields to be weak so as not to affect the assumption of axisymmetry of the turbulence. An imposed uniform magnetic field in the preferred direction would still be allowed, but this case will not be investigated in this paper; see Brandenburg et al. (2009) for numerical investigations of passive scalar transport with a uniform field.

Except for a few comparison cases, we always consider flows in a slab between stress-free boundaries. This is the simplest example of flows that are nonvanishing on the boundary and compatible with axisymmetric turbulence. To facilitate comparison with earlier work on forced turbulence, we consider an isothermal layer even in the density-stratified case, i.e., there is no convection, and the flow is driven by a prescribed random forcing, just as we did in the case of homogeneous turbulence. This is similar to earlier work on forced homogeneous turbulence, but now we will be able to address questions regarding vertical pumping as well as helicity production and α effect in the presence of rotation. This setup allows us to isolate effects of density stratification from those originating from the nonuniformities of turbulence intensity and local correlation length. In addition to isothermal stratification, we assume an isothermal equation of state and thus do not consider an equation for the specific entropy. Hence, no Brunt-Väisälä oscillations can occur. This assumption would need to be relaxed for studying turbulent convection, which will be the subject of a future investigation.

2. Mean-field concept in turbulent transport

2.1. Mean electromotive force

The evolution of the magnetic field \mathbf{B} in an electrically conducting fluid is assumed to obey the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B} - \eta \mathbf{J}), \quad (1)$$

where \mathbf{U} is the velocity and η the microscopic magnetic diffusivity of the fluid, and \mathbf{J} is defined by $\mathbf{J} = \nabla \times \mathbf{B}$ (so that \mathbf{J}/μ_0 with μ_0 being the magnetic permeability is the electric current density). We define mean fields as averages, assume that the averaging satisfies (exactly or approximately) the Reynolds rules, and denote averaged quantities by overbars.¹ The mean magnetic field $\overline{\mathbf{B}}$ is then governed by

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \overline{\boldsymbol{\mathcal{E}}} - \eta \overline{\mathbf{J}}), \quad (2)$$

where $\overline{\boldsymbol{\mathcal{E}}} = \overline{\mathbf{u} \times \mathbf{b}}$ is the mean electromotive force resulting from the correlation of velocity and magnetic field fluctuations, $\mathbf{u} = \mathbf{U} - \overline{\mathbf{U}}$ and $\mathbf{b} = \mathbf{B} - \overline{\mathbf{B}}$.

We focus attention on the mean electromotive force $\overline{\boldsymbol{\mathcal{E}}}$ in cases in which the velocity fluctuations \mathbf{u} constitute axisymmetric turbulence, that is, turbulence with one preferred direction, which we describe by the unit vector $\hat{\mathbf{e}}$. Until further notice we accept the traditional assumption according to which $\overline{\boldsymbol{\mathcal{E}}}$ in a given point in space and time is a linear homogeneous function of $\overline{\mathbf{B}}$ and its first spatial derivatives in this point. Then, $\overline{\boldsymbol{\mathcal{E}}}$ can be represented in the form

$$\begin{aligned} \overline{\boldsymbol{\mathcal{E}}} = & -\alpha_{\perp} \overline{\mathbf{B}} - (\alpha_{\parallel} - \alpha_{\perp})(\hat{\mathbf{e}} \cdot \overline{\mathbf{B}})\hat{\mathbf{e}} - \gamma \hat{\mathbf{e}} \times \overline{\mathbf{B}} \\ & -\beta_{\perp} \overline{\mathbf{J}} - (\beta_{\parallel} - \beta_{\perp})(\hat{\mathbf{e}} \cdot \overline{\mathbf{J}})\hat{\mathbf{e}} - \delta \hat{\mathbf{e}} \times \overline{\mathbf{J}} \\ & -\kappa_{\perp} \overline{\mathbf{K}} - (\kappa_{\parallel} - \kappa_{\perp})(\hat{\mathbf{e}} \cdot \overline{\mathbf{K}})\hat{\mathbf{e}} - \mu \hat{\mathbf{e}} \times \overline{\mathbf{K}} \end{aligned} \quad (3)$$

¹ The Reynolds rules imply that $\overline{\overline{F+G}} = \overline{F} + \overline{G}$, $\overline{\overline{F}} = \overline{F}$, $\overline{\overline{FG}} = \overline{FG}$, $\overline{\partial F/\partial x} = \partial \overline{F}/\partial x$ and $\overline{\partial F/\partial t} = \partial \overline{F}/\partial t$ for any fluctuating quantities F and G .

with nine coefficients α_{\perp} , α_{\parallel} , \dots , μ .² Like $\bar{\mathbf{J}} = \nabla \times \bar{\mathbf{B}}$, also $\bar{\mathbf{K}}$ is determined by the gradient tensor $\nabla \bar{\mathbf{B}}$. While $\bar{\mathbf{J}}$ is given by its antisymmetric part, $\bar{\mathbf{K}}$ is a vector defined by $\bar{\mathbf{K}} = \hat{\mathbf{e}} \cdot (\nabla \bar{\mathbf{B}})^S$ with $(\nabla \bar{\mathbf{B}})^S$ being the symmetric part of $\nabla \bar{\mathbf{B}}$. A more detailed explanation of (3) is given in Appendix A. If $\hat{\mathbf{e}}$ is understood as polar vector (for example $\nabla \bar{\rho} / |\nabla \bar{\rho}|$, where $\bar{\rho}$ is the mean mass density), then $\bar{\mathbf{K}}$ is axial and γ , β_{\perp} , β_{\parallel} and μ are true scalars, but α_{\perp} , α_{\parallel} , δ , κ_{\perp} and κ_{\parallel} pseudoscalars. (Scalars are invariant but pseudoscalars change sign if the turbulent velocity field is reflected at a point or at a plane containing the preferred axis.) Sometimes it is useful to interpret $\hat{\mathbf{e}}$ as an axial vector (for example $\boldsymbol{\Omega} / |\boldsymbol{\Omega}|$ with $\boldsymbol{\Omega}$ being an angular velocity). Then, $\bar{\mathbf{K}}$ is a polar vector, β_{\perp} , β_{\parallel} , δ , κ_{\perp} , κ_{\parallel} and μ are true scalars but α_{\perp} , α_{\parallel} and γ pseudoscalars.

We may split $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ into parts $\bar{\mathbf{E}}_{\perp}$ and $\bar{\mathbf{B}}_{\perp}$ perpendicular to $\hat{\mathbf{e}}$ and parts $\bar{\mathbf{E}}_{\parallel}$ and $\bar{\mathbf{B}}_{\parallel}$ parallel to it. Then (3) can be written in the form

$$\begin{aligned} \bar{\mathbf{E}}_{\perp} &= -\alpha_{\perp} \bar{\mathbf{B}}_{\perp} - \gamma \hat{\mathbf{e}} \times \bar{\mathbf{B}}_{\perp} - \beta_{\perp} \bar{\mathbf{J}}_{\perp} - \delta \hat{\mathbf{e}} \times \bar{\mathbf{J}}_{\perp} \\ &\quad - \kappa_{\perp} \bar{\mathbf{K}}_{\perp} - \mu \hat{\mathbf{e}} \times \bar{\mathbf{K}}_{\perp} \\ \bar{\mathbf{E}}_{\parallel} &= -\alpha_{\parallel} \bar{\mathbf{B}}_{\parallel} - \beta_{\parallel} \bar{\mathbf{J}}_{\parallel} - \kappa_{\parallel} \bar{\mathbf{K}}_{\parallel}. \end{aligned} \quad (4)$$

Let us return to (3). In the simple case of homogeneous isotropic turbulence we have $\alpha_{\perp} = \alpha_{\parallel}$ and $\beta_{\perp} = \beta_{\parallel}$, and all remaining coefficients vanish. Then, (3) takes the form $\bar{\mathbf{E}} = \alpha \bar{\mathbf{B}} - \eta_t \bar{\mathbf{J}}$ with properly defined α and η_t . These two coefficients have been determined by test-field calculations (Sur et al., 2008; Brandenburg et al., 2008a).

In several previous studies of $\bar{\mathbf{E}}$, more general kinds of turbulence (that is, not only axisymmetric turbulence) have been considered, but with a less general definition of mean fields, which were just horizontal averages. More precisely, Cartesian coordinates (x, y, z) were adopted and the averages were taken over all x and y so that they depend on z and t only (Brandenburg et al., 2008a,b). This definition implies remarkable simplifications. Of course, we then have $\bar{J}_z = 0$. Further, there are no non-zero components of $\nabla \bar{\mathbf{B}}$ other than $\bar{B}_{x,z}$ and $\bar{B}_{y,z}$, for $\nabla \cdot \bar{\mathbf{B}} = 0$ requires $\bar{B}_{z,z} = 0$, and these components can be expressed as components of $\bar{\mathbf{J}}$, *viz.* $\bar{B}_{x,z} = \bar{J}_y$ and $\bar{B}_{y,z} = -\bar{J}_x$. (Here and in what follows, commas denote partial derivatives.) This again implies $\bar{\mathbf{K}} = -\frac{1}{2} \hat{\mathbf{e}} \times \bar{\mathbf{J}}$. As a consequence, this definition of mean fields reduces (3) to

$$\begin{aligned} \bar{\mathbf{E}} &= -\alpha_{\perp} \bar{\mathbf{B}} - (\alpha_{\parallel} - \alpha_{\perp}) (\hat{\mathbf{e}} \cdot \bar{\mathbf{B}}) \hat{\mathbf{e}} - \gamma \hat{\mathbf{e}} \times \bar{\mathbf{B}} \\ &\quad - \beta^{\dagger} \bar{\mathbf{J}} - \delta^{\dagger} \hat{\mathbf{e}} \times \bar{\mathbf{J}}, \end{aligned} \quad (5)$$

where $\beta^{\dagger} = \beta_{\perp} + \frac{1}{2} \mu$ and $\delta^{\dagger} = \delta - \frac{1}{2} \kappa_{\perp}$. Of course, α_{\perp} , α_{\parallel} , γ , β^{\dagger} and δ^{\dagger} are independent of x or y . Clearly, β_{\perp} and μ as well as δ and κ_{\perp} have no longer independent meanings. From (2) we may conclude that $\partial \bar{B}_z / \partial t = 0$. If we restrict ourselves to applications in which \bar{B}_z vanishes initially, it does so at all times and the term with $\alpha_{\parallel} - \alpha_{\perp}$ in (5) disappears. Then, only the four coefficients α_{\perp} , γ , β^{\dagger} and δ^{\dagger} are of interest. They can be determined by test-field

² Note that the signs in front of some individual terms on the right-hand side of (3), in particular of those with α_{\perp} and α_{\parallel} (perpendicular and parallel α effect) as well as γ (pumping in the z direction), may differ from the signs used in other representations.

calculations using two test fields independent of x and y (Brandenburg et al., 2008a,b).

In this paper we go in several respect beyond the assumptions mentioned so far. Firstly, we relax the assumption that $\bar{\mathbf{E}}$ in a given point in space is a homogeneous function of $\bar{\mathbf{B}}$ and its first spatial derivatives in this point. Instead, we admit a non-local connection between $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$. For simplicity, however, we further on assume that $\bar{\mathbf{E}}$ at a given time depends only on $\bar{\mathbf{B}}$ at the same time, that is, we remain with an instantaneous connection between $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$. This approximation requires that the mean field varies slowly on a time scale much longer than the turnover time of the turbulence; see Hubbard & Brandenburg (2009) for a more general treatment of rapidly changing fields. Secondly, we consider mean fields no longer as averages over all x and y . We define $\bar{\mathbf{B}}$ at a point (x, y) in a plane $z = \text{const}$ by averaging over some surroundings of this point in this plane so that it still depends on x and y . In that sense we generalize (3) so that

$$\begin{aligned} \bar{\mathbf{E}}(\mathbf{x}) &= - \int (\alpha_{\perp}(\mathbf{x}, \boldsymbol{\xi}) \bar{\mathbf{B}}(\mathbf{x} - \boldsymbol{\xi}) \\ &\quad + (\alpha_{\parallel}(\mathbf{x}, \boldsymbol{\xi}) - \alpha_{\perp}(\mathbf{x}, \boldsymbol{\xi})) (\hat{\mathbf{e}} \cdot \bar{\mathbf{B}}(\mathbf{x} - \boldsymbol{\xi})) \hat{\mathbf{e}} \\ &\quad + \gamma(\mathbf{x}, \boldsymbol{\xi}) \hat{\mathbf{e}} \times \bar{\mathbf{B}}(\mathbf{x} - \boldsymbol{\xi}) \\ &\quad + \beta_{\perp}(\mathbf{x}, \boldsymbol{\xi}) \bar{\mathbf{J}}(\mathbf{x} - \boldsymbol{\xi}) \\ &\quad + (\beta_{\parallel}(\mathbf{x}, \boldsymbol{\xi}) - \beta_{\perp}(\mathbf{x}, \boldsymbol{\xi})) (\hat{\mathbf{e}} \cdot \bar{\mathbf{J}}(\mathbf{x} - \boldsymbol{\xi})) \hat{\mathbf{e}} \\ &\quad + \delta(\mathbf{x}, \boldsymbol{\xi}) \hat{\mathbf{e}} \times \bar{\mathbf{J}}(\mathbf{x} - \boldsymbol{\xi}) \\ &\quad + \kappa_{\perp}(\mathbf{x}, \boldsymbol{\xi}) \bar{\mathbf{K}}(\mathbf{x} - \boldsymbol{\xi}) \\ &\quad + (\kappa_{\parallel}(\mathbf{x}, \boldsymbol{\xi}) - \kappa_{\perp}(\mathbf{x}, \boldsymbol{\xi})) (\hat{\mathbf{e}} \cdot \bar{\mathbf{K}}(\mathbf{x} - \boldsymbol{\xi})) \hat{\mathbf{e}} \\ &\quad + \mu(\mathbf{x}, \boldsymbol{\xi}) \hat{\mathbf{e}} \times \bar{\mathbf{K}}(\mathbf{x} - \boldsymbol{\xi})) d^3 \boldsymbol{\xi}. \end{aligned} \quad (6)$$

As a consequence of the axisymmetry of the turbulence, the coefficients α_{\perp} , α_{\parallel} , \dots , μ depend only via $\xi_x^2 + \xi_y^2$ on ξ_x and ξ_y . We consider them also as symmetric in ξ_z . The integration is over all $\boldsymbol{\xi}$ space. Of course, $\bar{\mathbf{E}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{J}}$, and $\bar{\mathbf{K}}$ may depend on t . For simplicity, however, the argument t has been dropped.

Let us subject (6) to a Fourier transformation with respect to $\boldsymbol{\xi}$. We define it by

$$F(\boldsymbol{\xi}) = (2\pi)^{-3} \int \tilde{F}(\mathbf{k}) \exp(i\mathbf{k} \cdot \boldsymbol{\xi}) d^3 k. \quad (7)$$

Remembering the convolution theorem we obtain

$$\begin{aligned} \bar{\mathbf{E}}(\mathbf{x}) &= -(2\pi)^{-3} \int (\tilde{\alpha}_{\perp}(\mathbf{x}, \mathbf{k}) \tilde{\bar{\mathbf{B}}}(\mathbf{k}) \\ &\quad + (\tilde{\alpha}_{\parallel}(\mathbf{x}, \mathbf{k}) - \tilde{\alpha}_{\perp}(\mathbf{x}, \mathbf{k})) (\hat{\mathbf{e}} \cdot \tilde{\bar{\mathbf{B}}}(\mathbf{k})) \hat{\mathbf{e}} \\ &\quad + \tilde{\gamma}(\mathbf{x}, \mathbf{k}) \hat{\mathbf{e}} \times \tilde{\bar{\mathbf{B}}}(\mathbf{k}) \\ &\quad + \tilde{\beta}_{\perp}(\mathbf{x}, \mathbf{k}) \tilde{\bar{\mathbf{J}}}(\mathbf{k}) + (\tilde{\beta}_{\parallel}(\mathbf{x}, \mathbf{k}) - \tilde{\beta}_{\perp}(\mathbf{x}, \mathbf{k})) (\hat{\mathbf{e}} \cdot \tilde{\bar{\mathbf{J}}}(\mathbf{k})) \hat{\mathbf{e}} \\ &\quad + \tilde{\delta}(\mathbf{x}, \mathbf{k}) \hat{\mathbf{e}} \times \tilde{\bar{\mathbf{J}}}(\mathbf{k}) \\ &\quad + \tilde{\kappa}_{\perp}(\mathbf{x}, \mathbf{k}) \tilde{\bar{\mathbf{K}}}(\mathbf{k}) + (\tilde{\kappa}_{\parallel}(\mathbf{x}, \mathbf{k}) - \tilde{\kappa}_{\perp}(\mathbf{x}, \mathbf{k})) (\hat{\mathbf{e}} \cdot \tilde{\bar{\mathbf{K}}}(\mathbf{k})) \hat{\mathbf{e}} \\ &\quad + \tilde{\mu}(\mathbf{x}, \mathbf{k}) \hat{\mathbf{e}} \times \tilde{\bar{\mathbf{K}}}(\mathbf{k})) \exp(i\mathbf{k} \cdot \mathbf{x}) d^3 k; \end{aligned} \quad (8)$$

see Chatterjee et al. (2011) for a corresponding relation in the case of horizontally averaged magnetic fields that depend only on z . Like α_{\perp} , α_{\parallel} , \dots , μ , the $\tilde{\alpha}_{\perp}$, $\tilde{\alpha}_{\parallel}$, \dots , $\tilde{\mu}$ are real quantities. They depend only via $k_{\perp} = (k_x^2 + k_y^2)^{1/2}$

on k_x and k_y and are symmetric in k_z , i.e., depend only via $k_{\parallel} = |k_z|$ on k_z . Due to the reality of the α_{\perp} , α_{\parallel} , \dots , μ and their symmetry in ξ_x , ξ_y and ξ_z we have

$$\tilde{\alpha}_{\perp}(\mathbf{x}, \mathbf{k}) = \int \alpha_{\perp}(\mathbf{x}, \boldsymbol{\xi}) \cos k_x \xi_x \cos k_y \xi_y \cos k_z \xi_z d^3 \xi \quad (9)$$

and analogous relations for $\tilde{\alpha}_{\parallel}$, \dots , $\tilde{\mu}$. We note that $\tilde{\alpha}_{\perp}$, \dots , $\tilde{\mu}$, taken at $\mathbf{k} = 0$, agree with α_{\perp} , \dots , μ in Equation (3).

2.2. Mean passive scalar flux

There are interesting analogies between turbulent transport of magnetic flux and that of a passive scalar (cf. Rädler et al., 2011). Assume that the evolution of a passive scalar C , e.g., the concentration of an admixture in a fluid, is given by

$$\frac{\partial C}{\partial t} = -\nabla \cdot (\mathbf{U}C - D\nabla C), \quad (10)$$

where D is the microscopic (molecular) diffusivity. Then the mean scalar \bar{C} has to satisfy

$$\frac{\partial \bar{C}}{\partial t} = -\nabla \cdot (\bar{\mathbf{U}}\bar{C} + \bar{\mathcal{F}} - D\nabla \bar{C}), \quad (11)$$

where $\bar{\mathcal{F}} = \overline{\mathbf{u}c}$ is the mean passive scalar flux, \mathbf{u} stands again for the fluctuations of the velocity and $c = C - \bar{C}$ for the fluctuations of C . Consider again axisymmetric turbulence with a preferred direction given by the unit vector $\hat{\mathbf{e}}$. Assume that $\bar{\mathcal{F}}$ in a given point in space and time is determined by \bar{C} and its gradient $\bar{\mathbf{G}} = \nabla \bar{C}$ in this point. Then we have

$$\bar{\mathcal{F}} = -\gamma^C \bar{C} \hat{\mathbf{e}} - \beta_{\perp}^C \bar{\mathbf{G}} - (\beta_{\parallel}^C - \beta_{\perp}^C) (\hat{\mathbf{e}} \cdot \bar{\mathbf{G}}) \hat{\mathbf{e}} - \delta^C \hat{\mathbf{e}} \times \bar{\mathbf{G}}, \quad (12)$$

with coefficients γ^C , β_{\perp}^C , β_{\parallel}^C and δ^C . If $\hat{\mathbf{e}}$ is a polar vector, γ^C is a scalar but δ^C a pseudoscalar, and if $\hat{\mathbf{e}}$ is an axial vector, γ^C is a pseudoscalar but δ^C a scalar, while β_{\perp}^C and β_{\parallel}^C are always scalars. We note that $\nabla \cdot (\delta^C \hat{\mathbf{e}} \times \bar{\mathbf{G}})$ is only unequal zero if δ^C is not constant but varies in the direction of $\hat{\mathbf{e}} \times \bar{\mathbf{G}}$.

We may split $\bar{\mathcal{F}}$ and $\bar{\mathbf{G}}$ into parts $\bar{\mathcal{F}}_{\perp}$ and $\bar{\mathbf{G}}_{\perp}$ perpendicular to $\hat{\mathbf{e}}$, and parts $\bar{\mathcal{F}}_{\parallel}$ and $\bar{\mathbf{G}}_{\parallel}$ parallel to it, and give (12) the form

$$\begin{aligned} \bar{\mathcal{F}}_{\perp} &= -\beta_{\perp}^C \bar{\mathbf{G}}_{\perp} - \delta^C \hat{\mathbf{e}} \times \bar{\mathbf{G}}_{\perp} \\ \bar{\mathcal{F}}_{\parallel} &= -\gamma^C \hat{\mathbf{e}} \bar{C} - \beta_{\parallel}^C \bar{\mathbf{G}}_{\parallel}. \end{aligned} \quad (13)$$

Let us now relax the assumption that $\bar{\mathcal{F}}$ in a given point in space and time is determined by \bar{C} and $\bar{\mathbf{G}}$ in this point. Analogously to the magnetic case we consider a non-local but instantaneous connection between $\bar{\mathcal{F}}$ and \bar{C} . Then we have

$$\begin{aligned} \bar{\mathcal{F}}(\mathbf{x}) &= - \int \left(\gamma^C(\mathbf{x}, \boldsymbol{\xi}) \hat{\mathbf{e}} \bar{C}(\mathbf{x} - \boldsymbol{\xi}) \right. \\ &\quad + \beta_{\perp}^C(\mathbf{x}, \boldsymbol{\xi}) \bar{\mathbf{G}}(\mathbf{x} - \boldsymbol{\xi}) \\ &\quad + (\beta_{\parallel}^C(\mathbf{x}, \boldsymbol{\xi}) - \beta_{\perp}^C(\mathbf{x}, \boldsymbol{\xi})) (\hat{\mathbf{e}} \cdot \bar{\mathbf{G}}(\mathbf{x} - \boldsymbol{\xi})) \hat{\mathbf{e}} \\ &\quad \left. + \delta^C(\mathbf{x}, \boldsymbol{\xi}) \hat{\mathbf{e}} \times \bar{\mathbf{G}}(\mathbf{x} - \boldsymbol{\xi}) \right) d^3 \xi. \end{aligned} \quad (14)$$

As α_{\perp} , α_{\parallel} , \dots , μ in the magnetic case, γ^C , β_{\perp}^C , β_{\parallel}^C and δ^C depend only via $\xi_x^2 + \xi_y^2$ on ξ_x and ξ_y , and we consider them

also as symmetric in ξ_z . The integration is again over all $\boldsymbol{\xi}$ space. Note that $\bar{\mathcal{F}}$, \bar{C} , and $\bar{\mathbf{G}}$ may, even if it is not explicitly indicated, depend on t . Applying the Fourier transformation defined by (7) on (14), we arrive at

$$\begin{aligned} \bar{\mathcal{F}}(\mathbf{x}) &= -(2\pi)^{-3} \int \left(\tilde{\gamma}^C(\mathbf{x}, \mathbf{k}) \hat{\mathbf{e}} \tilde{\bar{C}}(\mathbf{k}) \right. \\ &\quad + \tilde{\beta}_{\perp}^C(\mathbf{x}, \mathbf{k}) \tilde{\bar{\mathbf{G}}}(\mathbf{k}) \\ &\quad + (\tilde{\beta}_{\parallel}^C(\mathbf{x}, \mathbf{k}) - \tilde{\beta}_{\perp}^C(\mathbf{x}, \mathbf{k})) (\hat{\mathbf{e}} \cdot \tilde{\bar{\mathbf{G}}}(\mathbf{k})) \hat{\mathbf{e}} \\ &\quad \left. + \tilde{\delta}^C(\mathbf{x}, \mathbf{k}) \hat{\mathbf{e}} \times \tilde{\bar{\mathbf{G}}}(\mathbf{k}) \right) \exp(i\mathbf{k} \cdot \mathbf{x}) d^3 k, \end{aligned} \quad (15)$$

where $\tilde{\gamma}_{\perp}^C$, $\tilde{\beta}_{\perp}^C$, $\tilde{\beta}_{\parallel}^C$ and $\tilde{\delta}^C$ are real quantities. They depend only via $k_x^2 + k_y^2$ on k_x and k_y , and only via k_{\parallel} on k_z , and they satisfy relations analogous to (9). We note that $\tilde{\gamma}^C$, $\tilde{\beta}_{\perp}^C$, $\tilde{\beta}_{\parallel}^C$, and $\tilde{\delta}^C$ at $\mathbf{k} = 0$ agree with γ^C , β_{\perp}^C , β_{\parallel}^C , and δ^C in (12).

3. Simulating the turbulence

We assume that the fluid is compressible and its flow is governed by the equations

$$\begin{aligned} \frac{D\mathbf{U}}{Dt} &= \mathbf{f} + \mathbf{g} - \nabla h - 2\boldsymbol{\Omega} \times \mathbf{U} + \rho^{-1} \nabla \cdot (2\nu \rho \mathbf{S}) \\ \frac{Dh}{Dt} &= -c_s^2 \nabla \cdot \mathbf{U}. \end{aligned} \quad (16)$$

Here, \mathbf{f} means a random force which primarily drives isotropic turbulence (e.g., Haugen et al., 2004), \mathbf{g} the gravitational force, and h the specific enthalpy. An isothermal equation of state, $p = \rho c_s^2$, has been adopted with a constant isothermal sound speed c_s . In general a fluid flow in a rotating system is considered, $\boldsymbol{\Omega}$ is the angular velocity which defines the Coriolis force. As usual ρ means the mass density, ν the kinematic viscosity and \mathbf{S} the trace-free rate of strain tensor, $S_{ij} = \frac{1}{2}(\partial U_{i,j} + \partial U_{j,i}) - \frac{1}{3}\delta_{ij} \nabla \cdot \mathbf{U}$. The influence of the magnetic field on the fluid motion, that is the Lorentz force, is ignored throughout the paper.

The numerical simulation is carried out in a cubic domain of size L^3 , so the smallest wavenumber is $k_1 = 2\pi/L$. In most of the cases a density stratification is included with $\mathbf{g} = (0, 0, -g)$, so the density scale height is $H_{\rho} = c_s^2/g$. The number of scale heights across the domain is equal to $\Delta \ln \rho$, where Δ denotes the difference of values at the two edges of the domain. Since $L/H_{\rho} = 2\pi$ we have a density contrast of $\exp 2\pi \approx 535$. The forcing is assumed to work with an average wavenumber k_f . The scale separation ratio is then given by k_f/k_1 , for which we usually adopt the value 5. This means that we have about 5 eddies in each of the three coordinate directions.

The flow inside the considered domain depends on the boundary conditions. Unless indicated otherwise we take the top and bottom surfaces $z = z_1$ and $z = z_2$ with $z_2 = -z_1 = L/2$ as stress-free and adopt periodic boundary conditions for the other surfaces.

4. Computing the transport coefficients

4.1. Test-field method

In the magnetic case the coefficients α_{\perp} , α_{\parallel} , \dots , μ are determined by the test-field method (Schirmer et al., 2005,

2007; Brandenburg et al., 2008a). This method works with a set of test fields $\overline{\mathbf{B}}$, called $\overline{\mathbf{B}}^T$, and the corresponding mean electromotive forces $\overline{\mathcal{E}}$, called $\overline{\mathcal{E}}^T$. For the latter we have $\overline{\mathcal{E}}^T = \overline{\mathbf{u}} \times \overline{\mathbf{b}}^T$, where the \mathbf{b}^T obey

$$\begin{aligned} \mathbf{b}^T &= \nabla \times \mathbf{a}^T \\ \frac{\partial \mathbf{a}^T}{\partial t} &= \overline{\mathbf{U}} \times \mathbf{b}^T + \mathbf{u} \times \overline{\mathbf{B}}^T + (\mathbf{u} \times \mathbf{b}^T)' + \eta \nabla^2 \mathbf{a}^T, \end{aligned} \quad (17)$$

with $\overline{\mathbf{U}}$ and \mathbf{u} taken from the solutions of (16). For the boundaries $z = \text{const}$ we choose conditions which correspond to an adjacent perfect conductor, for the x and y directions periodic boundary conditions.

We define four test fields by

$$\begin{aligned} \overline{\mathcal{E}}^{1s} &= (B_0 \, sx \, sy \, sz, 0, 0), & \overline{\mathcal{E}}^{1c} &= (B_0 \, sx \, sy \, cz, 0, 0) \\ \overline{\mathcal{E}}^{2s} &= (0, 0, B_0 \, sx \, sy \, sz), & \overline{\mathcal{E}}^{2c} &= (0, 0, B_0 \, sx \, sy \, cz) \end{aligned} \quad (18)$$

with a constant B_0 . Here and in what follows we use the abbreviations

$$\begin{aligned} sx &= \sin k_x x, & cx &= \cos k_x x \\ sy &= \sin k_y y, & cy &= \cos k_y y \\ sz &= \sin k_z z, & cz &= \cos k_z z. \end{aligned} \quad (19)$$

We recall that test-fields need not to be solenoidal (see Schirmer et al., 2005, 2007).

We denote the mean electromotive forces which correspond to the test fields (18) by $\overline{\mathcal{E}}^{1s}$, $\overline{\mathcal{E}}^{1c}$, $\overline{\mathcal{E}}^{2s}$, and $\overline{\mathcal{E}}^{2c}$. With the presentation (6) and relations like (9) we find

$$\begin{aligned} \overline{\mathcal{E}}_x^{1s} &= -B_0 (\tilde{\alpha}_\perp \, sx \, sy \, sz - (\tilde{\delta} - \frac{1}{2} \tilde{\kappa}_\perp) k_z \, sx \, sy \, cz) \\ \overline{\mathcal{E}}_y^{1s} &= -B_0 (\tilde{\gamma} \, sx \, sy \, sz + (\tilde{\beta}_\perp + \frac{1}{2} \tilde{\mu}) k_z \, sx \, sy \, cz) \\ \overline{\mathcal{E}}_z^{1s} &= B_0 \tilde{\beta}_\parallel k_y \, sx \, cy \, sz \\ \overline{\mathcal{E}}_x^{2s} &= -B_0 ((\tilde{\beta}_\perp - \frac{1}{2} \tilde{\mu}) k_y \, sx \, cy \, sz + (\tilde{\delta} + \frac{1}{2} \tilde{\kappa}_\perp) k_x \, cx \, sy \, sz) \\ \overline{\mathcal{E}}_y^{2s} &= B_0 ((\tilde{\beta}_\perp - \frac{1}{2} \tilde{\mu}) k_x \, cx \, sy \, sz - (\tilde{\delta} + \frac{1}{2} \tilde{\kappa}_\perp) k_y \, sx \, cy \, sz) \\ \overline{\mathcal{E}}_z^{2s} &= -B_0 (\tilde{\alpha}_\parallel \, sx \, sy \, sz + \tilde{\kappa}_\parallel k_z \, sx \, sy \, cz) \end{aligned} \quad (20)$$

and corresponding relations for $\overline{\mathcal{E}}_x^{1c}, \dots, \overline{\mathcal{E}}_z^{2c}$, whose right-hand sides can be derived from those in (20) simply by replacing sz and cz by cz and $-sz$, respectively.

In view of the assumed axisymmetry of the turbulence, we consider $\alpha_\perp, \alpha_\parallel, \dots, \mu$ in what follows as independent of x and y but admit a dependence on z . When multiplying both sides of the equations (20) and of the corresponding ones for $\overline{\mathcal{E}}_x^{1c}, \dots, \overline{\mathcal{E}}_z^{2c}$ with $sx \, sy$, $sx \, cy$ or $cysy$ and averaging over all x and y , we obtain a system of equations, which can be solved for $\tilde{\alpha}_\perp, \tilde{\alpha}_\parallel, \dots, \tilde{\mu}$. The result reads

$$\begin{aligned} \tilde{\alpha}_\perp &= -\langle b^{ss} (sz \overline{\mathcal{E}}_x^{1s} + cz \overline{\mathcal{E}}_x^{1c}) \rangle \\ \tilde{\alpha}_\parallel &= -\langle b^{ss} (sz \overline{\mathcal{E}}_z^{2s} + cz \overline{\mathcal{E}}_z^{2c}) \rangle \\ \tilde{\gamma} &= -\langle b^{ss} (sz \overline{\mathcal{E}}_y^{1s} + cz \overline{\mathcal{E}}_y^{1c}) \rangle \\ \tilde{\beta}_\perp &= -\frac{1}{2} \langle B^{ss} (cz \overline{\mathcal{E}}_y^{1s} - sz \overline{\mathcal{E}}_y^{1c}) + B^{sc} (sz \overline{\mathcal{E}}_x^{2s} + cz \overline{\mathcal{E}}_x^{2c}) \rangle \\ &= -\frac{1}{2} \langle B^{ss} (cz \overline{\mathcal{E}}_y^{1s} - sz \overline{\mathcal{E}}_y^{1c}) - B^{cs} (sz \overline{\mathcal{E}}_y^{2s} + cz \overline{\mathcal{E}}_y^{2c}) \rangle \\ \tilde{\beta}_\parallel &= \langle B^{sc} (sz \overline{\mathcal{E}}_z^{1s} + cz \overline{\mathcal{E}}_z^{1c}) \rangle \\ \tilde{\delta} &= \frac{1}{2} \langle B^{ss} (cz \overline{\mathcal{E}}_x^{1s} - sz \overline{\mathcal{E}}_x^{1c}) - B^{cs} (sz \overline{\mathcal{E}}_x^{2s} + cz \overline{\mathcal{E}}_x^{2c}) \rangle \end{aligned} \quad (21)$$

$$\begin{aligned} &= \frac{1}{2} \langle B^{ss} (cz \overline{\mathcal{E}}_x^{1s} - sz \overline{\mathcal{E}}_x^{1c}) - B^{sc} (sz \overline{\mathcal{E}}_y^{2s} + cz \overline{\mathcal{E}}_y^{2c}) \rangle \\ \tilde{\kappa}_\perp &= -\langle B^{ss} (cz \overline{\mathcal{E}}_x^{1s} - sz \overline{\mathcal{E}}_x^{1c}) + B^{cs} (sz \overline{\mathcal{E}}_x^{2s} + cz \overline{\mathcal{E}}_x^{2c}) \rangle \\ &= -\langle B^{ss} (cz \overline{\mathcal{E}}_x^{1s} - sz \overline{\mathcal{E}}_x^{1c}) + B^{sc} (sz \overline{\mathcal{E}}_y^{2s} + cz \overline{\mathcal{E}}_y^{2c}) \rangle \\ \tilde{\kappa}_\parallel &= -\langle B^{ss} (cz \overline{\mathcal{E}}_z^{2s} - sz \overline{\mathcal{E}}_z^{2c}) \rangle \\ \tilde{\mu} &= -\langle B^{ss} (cz \overline{\mathcal{E}}_y^{1s} - sz \overline{\mathcal{E}}_y^{1c}) - B^{sc} (sz \overline{\mathcal{E}}_x^{2s} + cz \overline{\mathcal{E}}_x^{2c}) \rangle \\ &= -\langle B^{ss} (cz \overline{\mathcal{E}}_y^{1s} - sz \overline{\mathcal{E}}_y^{1c}) + B^{cs} (sz \overline{\mathcal{E}}_y^{2s} + cz \overline{\mathcal{E}}_y^{2c}) \rangle, \end{aligned}$$

where

$$\begin{aligned} b^{ss} &= 4 \, sx \, sy / B_0, & B^{ss} &= h^{ss} / k_z \\ B^{cs} &= 4 \, cx \, sy / k_x B_0, & B^{sc} &= 4 \, sx \, cy / k_y B_0. \end{aligned} \quad (22)$$

The angle brackets indicate averaging over x and y . Although the relations (21) and (22) contain k_x, k_y and k_z as independent variables, the $\tilde{\alpha}_\perp, \tilde{\alpha}_\parallel, \dots, \tilde{\mu}$ should vary only via $k_\perp = (k_x^2 + k_y^2)^{1/2}$ with k_x and k_y , and only via k_\parallel with k_z .

4.2. Test-scalar method

In the passive-scalar case the coefficients $\gamma^C, \beta_\perp^C, \beta_\parallel^C$, and δ^C are determined by the test-scalar method with test scalars \overline{C}^T and the corresponding fluxes $\overline{\mathcal{F}}^T$. For the latter, we have $\overline{\mathcal{F}}^T = \overline{\mathbf{u}} c^T$, where c^T obeys

$$\frac{\partial c^T}{\partial t} = -\nabla \cdot (\overline{\mathbf{U}} c^T + \mathbf{u} \overline{C}^T + (\mathbf{u} c^T)' - D \nabla c^T). \quad (23)$$

Again $\overline{\mathbf{U}}$ and \mathbf{u} are taken from the solutions of (16).

We define two test-scalars \overline{C}^{Ts} and \overline{C}^{Tc} by

$$\overline{C}^s = C_0 \, sx \, sy \, sz, \quad \overline{C}^c = C_0 \, sx \, sy \, cz, \quad (24)$$

where C_0 is a constant and the abbreviations (19) are used. From (14) we then have

$$\begin{aligned} \overline{\mathcal{F}}_x^s &= -C_0 (\tilde{\beta}_\perp^C k_x \, cx \, sy \, sz - \tilde{\delta}^C k_y \, sx \, cy \, sz) \\ \overline{\mathcal{F}}_y^s &= -C_0 (\tilde{\beta}_\perp^C k_y \, sx \, cy \, sz + \tilde{\delta}^C k_x \, cx \, sy \, sz) \\ \overline{\mathcal{F}}_z^s &= -C_0 (\tilde{\gamma}_\perp^C \, sx \, sy \, sz + \tilde{\beta}_\parallel^C k_z \, sx \, sy \, cz) \end{aligned} \quad (25)$$

and analogous relations for $\overline{\mathcal{F}}_x^c, \dots, \overline{\mathcal{F}}_z^c$ with sz and cz replaced by cz and $-sz$, respectively.

Analogous to the magnetic case, we assume that $\gamma^C, \beta_\perp^C, \beta_\parallel^C$, and δ^C are independent of x and y but may depend on z . Analogous to (21) we find here

$$\begin{aligned} \tilde{\gamma}^C &= -\langle c^{ss} (sz \overline{\mathcal{F}}_z^s + cz \overline{\mathcal{F}}_z^c) \rangle \\ \tilde{\beta}_\perp^C &= -\langle C^{cs} (sz \overline{\mathcal{F}}_x^s + cz \overline{\mathcal{F}}_x^c) \rangle = -\langle C^{sc} (sz \overline{\mathcal{F}}_y^s + cz \overline{\mathcal{F}}_y^c) \rangle \\ \tilde{\beta}_\parallel^C &= -\langle C^{ss} (cz \overline{\mathcal{F}}_z^s - sz \overline{\mathcal{F}}_z^c) \rangle \\ \tilde{\delta}^C &= \langle C^{sc} (sz \overline{\mathcal{F}}_x^s + cz \overline{\mathcal{F}}_x^c) \rangle = -\langle C^{cs} (sz \overline{\mathcal{F}}_y^s + cz \overline{\mathcal{F}}_y^c) \rangle, \end{aligned} \quad (26)$$

where c^{ss}, C^{ss}, C^{sc} , and C^{cs} are defined like b^{ss}, B^{ss}, B^{sc} , and B^{cs} , with C_0 at the place of B_0 . The angle brackets indicate again averaging over x and y . Note that $\tilde{\gamma}^C, \tilde{\beta}_\perp^C, \tilde{\beta}_\parallel^C$, and $\tilde{\delta}^C$ should depend only via $k_\perp = (k_x^2 + k_y^2)^{1/2}$ on k_x and k_y , and only via k_\parallel on k_z .

4.3. Validation using the Roberts flow

For a validation of our test-field procedure for the determination of the coefficients occurring in (3) we rely on the Roberts flow. We define it here by

$$\mathbf{u} = u_0(-\cos k_0 x \sin k_0 y, \sin k_0 x \cos k_0 y, 2f \cos k_0 x \cos k_0 y), \quad (27)$$

with some wavenumber k_0 and a factor f which characterizes the ratio of the magnitude of u_z to that of u_x and u_y . We further define mean fields as averages over x and y with an averaging scale which is much larger than the period length $2\pi/k_0$ of the flow pattern. When calculating the mean electromotive force $\overline{\mathcal{E}}$ for this flow, we assume that it is a linear homogeneous function of $\overline{\mathbf{B}}$ and its first spatial derivatives and adopt the second-order correlation approximation. Although the Roberts flow is far from being axisymmetric, the result for $\overline{\mathcal{E}}$ can be written in the form (3), and we have

$$\begin{aligned} \alpha_{\perp} &= \frac{u_0^2 f}{2\eta k_0}, & \alpha_{\parallel} &= \gamma = 0 \\ \beta_{\perp} &= \frac{u_0^2(1+4f^2)}{16\eta k_0^2}, & \beta_{\parallel} &= \frac{u_0^2}{8\eta k_0^2}, & \delta &= 0 \\ \kappa_{\perp} &= \kappa_{\parallel} = 0, & \mu &= -\frac{u_0^2(1-4f^2)}{8\eta k_0^2} = 2(\beta_{\perp} - \beta_{\parallel}). \end{aligned} \quad (28)$$

It agrees with and can be deduced from results reported in Rädler et al. (2002a,b). As for the passive scalar case, an analogous analytical calculation of the mean scalar flow $\overline{\mathcal{F}}$ leads to (12) with

$$\gamma^C = 0, \quad \beta_{\perp}^C = \frac{u_0^2}{8Dk_0^2}, \quad \beta_{\parallel}^C = \frac{u_0^2 f^2}{2Dk_0^2}, \quad \delta^C = 0. \quad (29)$$

We may proceed from the local connection of $\overline{\mathcal{E}}$ with $\overline{\mathbf{B}}$ and its derivatives considered in (3) to the non-local ones given by (6) or (8). As a consequence of the deviation of the flow from axisymmetry, we can then no longer justify that coefficients like $\alpha_{\perp}(\boldsymbol{\xi})$ depend only via $\xi_x^2 + \xi_y^2$ on ξ_x and ξ_y , and coefficients like $\tilde{\alpha}_{\perp}(\mathbf{k})$ only via k_{\perp} on k_x and k_y . This applies analogously to the connection of $\overline{\mathcal{F}}$ with $\overline{\mathbf{C}}$ and its derivatives and to coefficients like $\beta_{\perp}(\boldsymbol{\xi})$ and $\tilde{\beta}_{\perp}(\mathbf{k})$.

A test-field calculation of the coefficients $\tilde{\alpha}_{\perp}, \tilde{\alpha}_{\parallel}, \dots, \tilde{\mu}$, as well as $\tilde{\gamma}^C, \dots, \tilde{\delta}^C$, has been carried out under the conditions of the second-order correlation approximation with \mathbf{u} given by (27) and $f = 1/\sqrt{2}$. Figure 1 shows the results obtained for $\tilde{\alpha}_{\perp}, \tilde{\beta}_{\perp}, \tilde{\beta}_{\parallel}$ and $\tilde{\mu}$, as well as $\tilde{\beta}_{\perp}^C$ and $\tilde{\beta}_{\parallel}^C$, as functions of k_{\perp}/k_f , with $k_f = \sqrt{2}k_0$, for two fixed ratios k_{\parallel}/k_{\perp} . In the limit $k_{\perp}/k_f \ll 1$ these coefficients take just the values of $\alpha_{\perp}, \beta_{\perp}, \beta_{\parallel}, \mu, \beta_{\perp}^C$ and β_{\parallel}^C given in (28) and (29). For larger values of k_{\perp}/k_f , as to be expected, the $\tilde{\alpha}_{\perp}, \tilde{\beta}_{\perp}, \tilde{\beta}_{\parallel}, \tilde{\mu}, \tilde{\beta}_{\perp}^C$ and $\tilde{\beta}_{\parallel}^C$ depend also on the ratio of k_x and k_y .

4.4. Dimensionless parameters and related issues

Within the framework of this paper, the coefficients $\alpha_{\perp}, \alpha_{\parallel}, \dots, \mu$ as well as $\tilde{\alpha}_{\perp}, \tilde{\alpha}_{\parallel}, \dots, \tilde{\mu}$, and likewise $\gamma^C, \beta_{\perp}^C, \dots, \delta^C$ and $\tilde{\gamma}^C, \tilde{\beta}_{\perp}^C, \dots, \tilde{\delta}^C$, have to be considered as functions

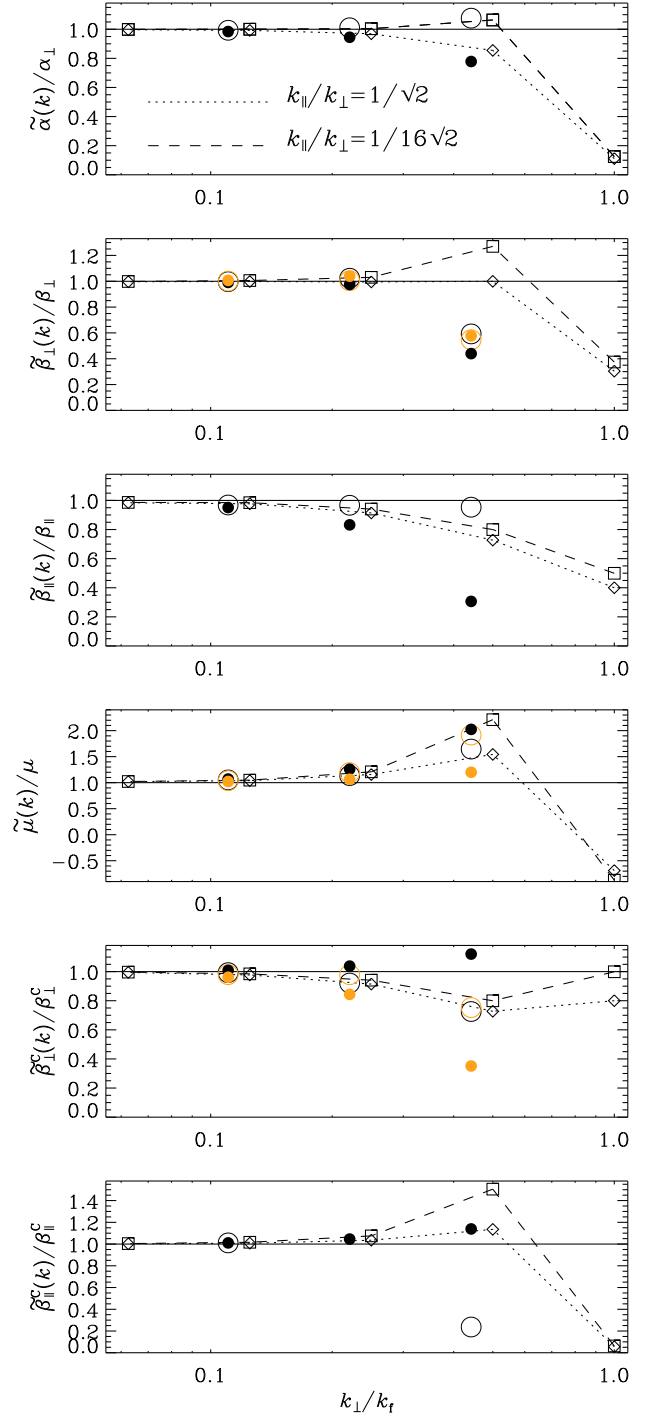


Fig. 1. The coefficients $\tilde{\alpha}_{\perp}, \tilde{\beta}_{\perp}, \tilde{\beta}_{\parallel}$, and $\tilde{\mu}$, as well as $\tilde{\beta}_{\perp}^C$ and $\tilde{\beta}_{\parallel}^C$ for the Roberts flow, calculated in the second-order correlation approximation, as functions of k_{\perp}/k_f , where $k_f = \sqrt{2}k_0$ is the effective wavenumber of the flow. Results obtained with $k_x = k_y$ and $k_{\parallel}/k_{\perp} = 1/\sqrt{2} \approx 0.7$ or $k_{\parallel}/k_{\perp} = 1/16\sqrt{2} \approx 0.004$ are represented by open squares and dotted lines or by open diamonds and dashed lines, respectively. Results with $k_x/k_y = 0.75$ [$\mathbf{k}_{\perp} = (3, 4, 0)k_1$] or $k_x/k_y = 5$ [$\mathbf{k}_{\perp} = (5, 1, 0)k_1$] and $k_{\parallel}/k_{\perp} = 0.2$ are indicated by open or filled circles, respectively. Orange and grey symbols correspond to the first and second expressions for $\tilde{\beta}_{\perp}$ and $\tilde{\mu}$ in (21) or for $\tilde{\beta}_{\parallel}^C$ in (26).

of several dimensionless parameters. In the magnetic case these are the magnetic Reynolds number $R_m = u_{\text{rms}}/\eta k_f$

and the magnetic Prandtl number $P_m = \nu/\eta$, in the passive scalar case the Péclet number $Pe = u_{\text{rms}}/Dk_f$ and the Schmidt number $Sc = \nu/D$, further the Mach number $Ma = u_{\text{rms}}/c_s$, the gravity parameter $Gr = g/c_s^2 k_f$, the Coriolis number $Co = 2\Omega/u_{\text{rms}}k_f$, as well as the scale separation ratio k_f/k_1 .

Throughout the rest of the paper we give the coefficients α_\perp , α_\parallel , γ , and γ^C as well as $\tilde{\alpha}_\perp$, $\tilde{\alpha}_\parallel$, $\tilde{\gamma}$, and $\tilde{\gamma}^C$ in units of $u_{\text{rms}}/3$, the remaining coefficients β_\perp , \dots , δ^C and $\tilde{\beta}_\perp$, \dots , $\tilde{\delta}^C$ in units of $u_{\text{rms}}/3k_f$. The numerical calculations deliver these coefficients as functions of z and t . To avoid boundary effects, we average these results over $-2 \leq k_1 z \leq 1$ (see Figure 3 below). The resulting time series are averaged over a range where the results are statistically stationary, i.e., there is no trend in the time series. Error bars are defined by comparing the maximum departure of an average over any one third of the time series with the full time average.

In the case of isotropic turbulence it has been observed that many of the transport coefficients enter an asymptotic regime as soon as R_m exceeds unity (Sur et al., 2008). While this should be checked in every new case again (see below), it is important to realize that, according to several earlier results (see also Brandenburg et al., 2009), only values of R_m below unity are characteristic of the diffusively dominated regime, while for R_m exceeding unity the transport coefficients turn out to be nearly independent of the value of R_m .

We are often interested in the limit $k_\perp, k_\parallel \rightarrow 0$, in which the $\tilde{\alpha}_\perp, \tilde{\alpha}_\parallel, \dots, \tilde{\delta}^C$ turn into the $\alpha_\perp, \alpha_\parallel, \dots, \delta^C$. In this limit, however, the test fields and test scalars defined by (18) and (24) vanish. Unless specified otherwise, we approach this limit by choosing the smallest possible non-zero $|k_x|$, $|k_y|$ and $|k_z|$, that is, by putting $k_x = k_y = k_z = k_1$.

In the figures of the next section results for $\tilde{\alpha}_\perp, \tilde{\alpha}_\parallel, \dots, \tilde{\delta}^C$ are represented. In all cases in which they are considered as results for the limit $k_\perp, k_\parallel \rightarrow 0$ they are simply denoted as $\alpha_\perp, \alpha_\parallel, \dots, \delta^C$ in the text.

5. Results

5.1. Homogeneous rotating turbulence

Let us first consider homogeneous turbulence in a rotating system, that is, under the influence of the Coriolis force. The angular velocity $\mathbf{\Omega}$ responsible for this force defines the preferred direction of the turbulence, $\hat{e} = \mathbf{\Omega}/|\mathbf{\Omega}|$. In this case we expect only contributions to the mean electromotive force $\overline{\mathcal{E}}$ from a spatially varying mean magnetic field $\overline{\mathbf{B}}$, and contributions to the passive scalar flux $\overline{\mathcal{F}}$ from a spatially varying mean passive scalar \overline{C} . That is, in (3) we have only the terms with $\beta_\perp, \beta_\parallel, \delta, \kappa_\perp, \kappa_\parallel$, and μ , and in (12) only those with $\beta_\perp^C, \beta_\parallel^C$, and δ^C . The terms with β_\perp and β_\parallel , as well as those with β_\perp^C and β_\parallel^C , characterize anisotropic mean-field diffusivities, and that with δ corresponds to the “ $\mathbf{\Omega} \times \overline{\mathbf{J}}$ effect” (Rädler, 1969a,b, 1976; Krause & Rädler, 1971, 1980), while the δ^C term vanishes underneath the divergence and is therefore without interest.

Figure 2 shows the dependence of the aforementioned coefficients on Co for $R_m \approx Pe \approx 9$ and $k_f/k_1 = 5$. The values of $\beta_\perp, \beta_\parallel, \beta_\perp^C$ and β_\parallel^C , which remain finite for $Co \rightarrow 0$, are always close together. The other four coefficients vary linearly with Co as long as Co is small. Specifically, we find

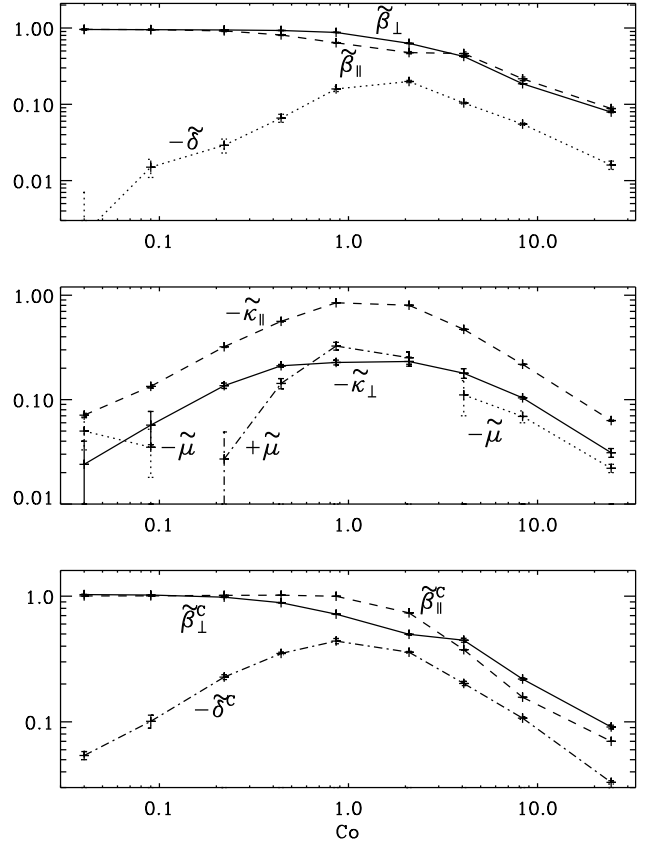


Fig. 2. Co dependence of transport coefficients in a model with rotation but zero density stratification, $R_m \approx 9$, $P_m = Sc = 1$, $Gr = 0$, $k_f/k_1 = 5$.

$\tilde{\delta} \approx -0.1 Co$, $\tilde{\delta}^C \approx -Co$, as well as $\tilde{\kappa}_\perp \approx -0.3 Co$ and $\tilde{\kappa}_\parallel \approx -Co$. These coefficients reach maxima at $Co \approx 1$. For rapid rotation, $|Co| \gg 1$, all coefficients approach zero like $1/Co$. In particular, we have $\beta_\perp \approx 1.2/Co$ and the same for $\beta_\parallel, \beta_\perp^C$, and β_\parallel^C , further $\tilde{\kappa}_\perp \approx -0.5/Co$, $\tilde{\kappa}_\parallel \approx -1.2/Co$, $\tilde{\delta} \approx -0.3/Co$, and $\tilde{\delta}^C \approx -0.6/Co$. Furthermore, we find that, within error bars, $\alpha_\perp, \alpha_\parallel, \gamma$, and γ^C are indeed zero.

5.2. Stratified turbulence

Owing to the presence of boundary conditions at the top and bottom of our domain and the lack of scale separation for our default choice of $k_f/k_1 = 5$, the turbulence is in all cases anisotropic, even if gravity is negligible. The ratio of the vertical and horizontal velocity components, $2\overline{u_\parallel^2}/\overline{u_\perp^2}$, is no longer, as in the isotropic case, equal to unity. For moderate stratification ($g/c_s^2 k_1 \approx 1$), not too large $|z|$, and $k_f/k_1 = 5$, it takes a value of about 0.9. It decreases when the ratio k_f/k_1 is decreased; see Table 1. Figure 3 shows the z dependence of $2\overline{u_\parallel^2}/\overline{u_\perp^2}$. For strong stratification and a high degree of scale separation, e.g. $k_f/k_1 = 30$, the mentioned ratio comes close to unity. Note, however, that smaller values of $2\overline{u_\parallel^2}/\overline{u_\perp^2}$ can be achieved in the non-isothermal case when the effects of buoyancy become important.

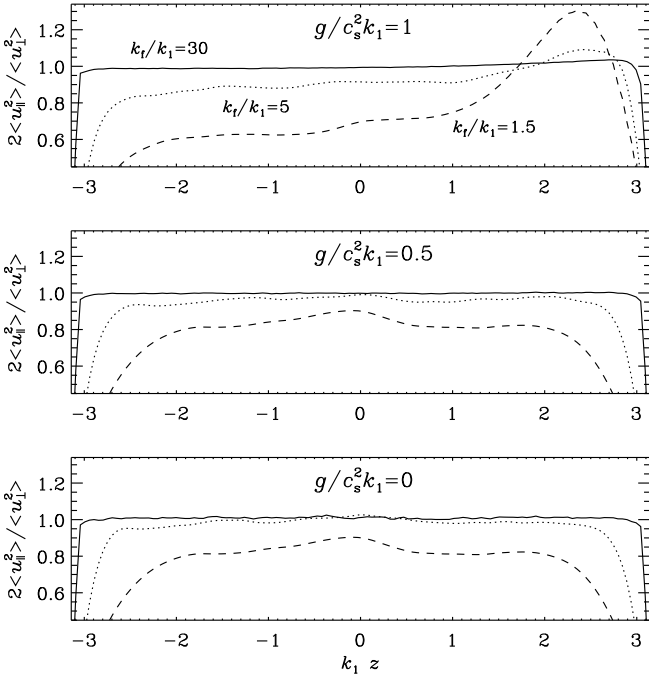


Fig. 3. Anisotropy $2\overline{u_{\parallel}^2}/\overline{u_{\perp}^2}$ of nonrotating turbulence for different stratifications, $g/c_s^2 k_1$, and different degrees of scale separation, k_f/k_1 .

5.2.1. Stratified nonrotating turbulence

For axisymmetric turbulence in a nonrotating system showing any kind of stratification in the representation (3) of $\overline{\mathcal{E}}$ only the four coefficients γ , β_{\perp} , β_{\parallel} , and μ can be non-zero. Likewise, in the representation (12) of $\overline{\mathcal{F}}$ only the three coefficients γ^C , β_{\perp}^C , and β_{\parallel}^C can be non-zero. Figure 4 shows their dependence on Gr. It appears that γ is always close to zero, while γ^C shows a linear increase for not too strong gravity. At the same time, β_{\perp} , β_{\parallel} , β_{\perp}^C , and β_{\parallel}^C remain approximately constant. We find that μ is negative and its modulus is mildly increasing with increasing stratification, but the error bars are large.

5.2.2. Stratified rotating turbulence

For turbulence under the influence of gravity and rotation, all nine coefficients α_{\perp} , \dots , μ are in general non-zero, as well as all four coefficients γ^C , \dots , δ^C . If both gravity and

Table 1. Dependence of the density contrast $\rho_{\text{bot}}/\rho_{\text{top}}$ and the degree of anisotropy $2\overline{u_{\parallel}^2}/\overline{u_{\perp}^2}$, for three different values of k_f/k_1 , on the density stratification $g/c_s^2 k_1$ for nonrotating turbulence. The values of $2\overline{u_{\parallel}^2}/\overline{u_{\perp}^2}$ have been obtained as averages over the range $-2 \leq k_1 z \leq 1$.

$g/c_s^2 k_1$	$\rho_{\text{bot}}/\rho_{\text{top}}$	$2\overline{u_{\parallel}^2}/\overline{u_{\perp}^2}$		
		$k_f = 1.5k_1$	$k_f = 5k_1$	$k_f = 30k_1$
0	0	0.84	0.99	1.00
0.5	23	0.84	0.97	1.00
1	540	0.66	0.90	0.99

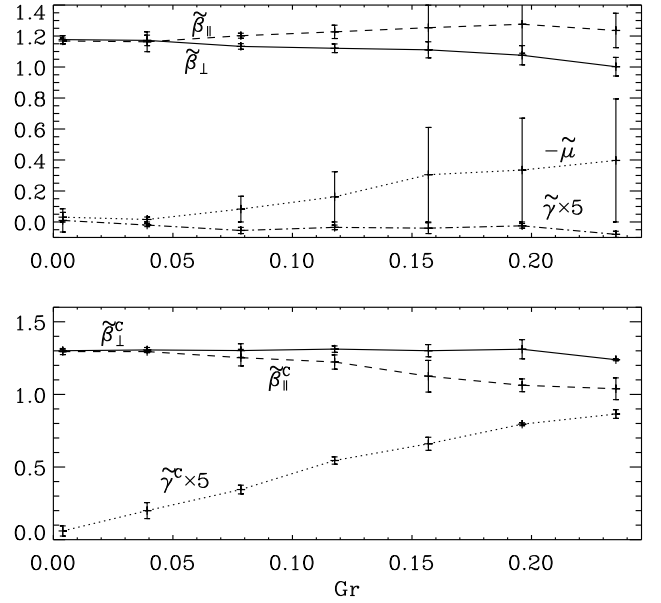


Fig. 4. Gr dependence of the transport coefficients in a model with density stratification but zero rotation, $P_m = Sc = 1$, $R_m \approx 22$, $Co = 0$, $k_f/k_1 = 5$.

rotation are so small that $\overline{\mathcal{E}}$ is linear in g and Ω , more precisely $\overline{\mathcal{E}}$ contains $g^m \Omega^n$, where n and m mean integers, only with $n + m \leq 1$, α_{\perp} and α_{\parallel} vanish but γ , β_{\perp} , δ and κ_{\perp} may well be unequal to zero. If $n + m \leq 2$, all nine coefficients may indeed be non-zero.

Results for stratified rotating turbulence are shown in Figure 5. The error bars are now bigger than either with just rotation or just stratification. For $Co \rightarrow 0$, the coefficients β_{\perp} , β_{\parallel} , μ , β_{\perp}^C , β_{\parallel}^C and δ^C remain finite. As Co is increased, their moduli show some decline. On the other hand the moduli of α_{\perp} , α_{\parallel} , γ , δ , κ_{\perp} , κ_{\parallel} and γ^C increase with Co as long as it is smaller than some value below unity but decrease again for larger Co . Both α_{\perp} and α_{\parallel} are negative, which is expected for \mathbf{g} and Ω being antiparallel to each other. Interestingly, μ is finite for small values of Co , in agreement with the result when there is only stratification (Figure 4), but with a modest amount of rotation, μ is suppressed and grows only when Co has reached values around unity.

5.3. Wavenumber dependence

So far we have considered the coefficients $\tilde{\alpha}_{\perp}$, $\tilde{\alpha}_{\parallel}$, \dots , $\tilde{\delta}^C$ in the limit $k = |\mathbf{k}| \rightarrow 0$, that is, $k_{\perp}, k_{\parallel} \rightarrow 0$. However, their behavior for larger k , in particular for k up to several k_f , is of interest, too. Most of them decrease like k^{-2} as k grows and can be fitted to a Lorentzian profile, as has been found in earlier calculation using the test-field method; see Brandenburg et al. (2008a), where in fact the dependence on k_{\parallel} was considered. Even earlier work that was not based on the test-field method showed a declining trend (Miesch et al., 2000; Brandenburg & Sokoloff, 2002). Nevertheless, as is shown in Figure 6, there are also some coefficients that first increase with k_{\parallel} , have a maximum near $k_{\parallel} = k_f$ and only then decrease with growing k_{\parallel} . Examples for such a

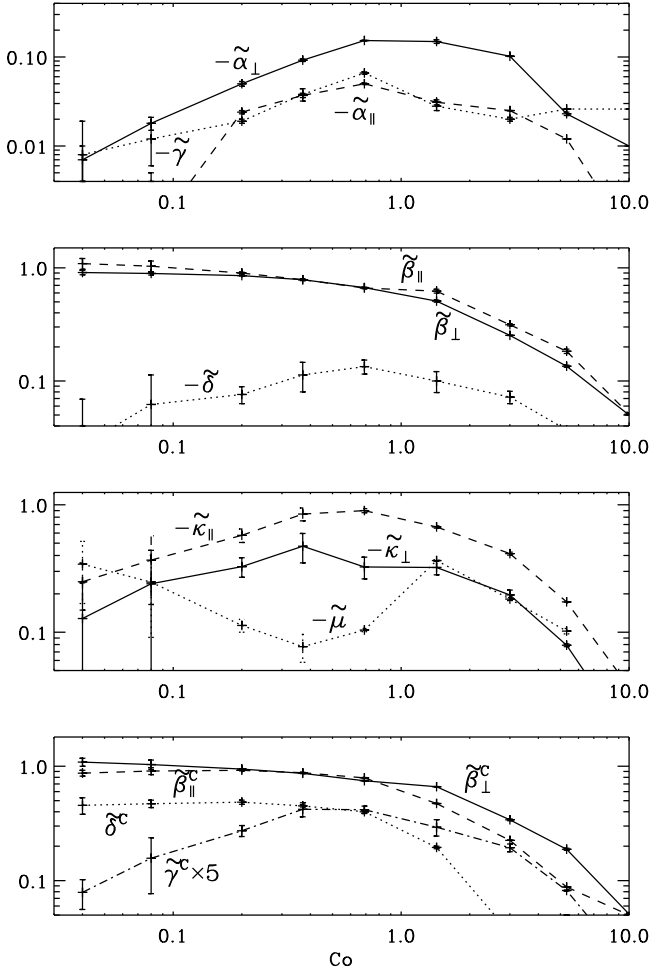


Fig. 5. Co dependence of transport coefficients in a model with rotation and density stratification, $P_m = Sc = 1$, $R_m \approx 10$, $Gr \approx 0.16$, $k_f/k_1 = 5$.

behavior are $\tilde{\alpha}_\parallel$, $\tilde{\delta}$, and $\tilde{\kappa}_\perp$, while $\tilde{\kappa}_\parallel$ peaks slightly below $k_\parallel = 0.5k_f$.

The dependence of the coefficients under discussion on k_\perp is shown in Figure 7. Note that our test fields vanish for $k_\perp = 0$, so no values are shown for this case. Note also that $-\tilde{\alpha}_\parallel$, $-\tilde{\delta}$, $-\tilde{\kappa}_\parallel$, and $-\tilde{\beta}_\perp$, which all have maxima for $k_\parallel/k_f \approx 1$, show a clear monotonic decline with k_\perp . Only $-\tilde{\kappa}_\perp$ has a maximum for intermediate values of both k_\parallel/k_f and k_\perp/k_f .

Most of the results presented in Figure 7 have been calculated with $k_x = k_y$, a few single ones for $\tilde{\alpha}_\perp$, $\tilde{\beta}_\perp$, $\tilde{\kappa}_\perp$ and $\tilde{\beta}_\perp^C$ also with $k_x/k_y = 0.75$ and $k_x/k_y = 0.2$. While the results for $\tilde{\beta}_\perp$ and $\tilde{\beta}_\perp^C$ agree well for all these values of k_x/k_y , there are significant discrepancies with $\tilde{\alpha}_\perp$ and $\tilde{\kappa}_\perp$.

5.4. Dependencies on R_m and Pe

Let us finally consider the dependence of all transport coefficients on R_m or Pe for a case where they are all expected to be finite. Therefore we choose again the case with $Co = 1$ and $Gr = 0.16$, which was also considered in Figures 5–7, and keep $P_m = Sc = 1$.

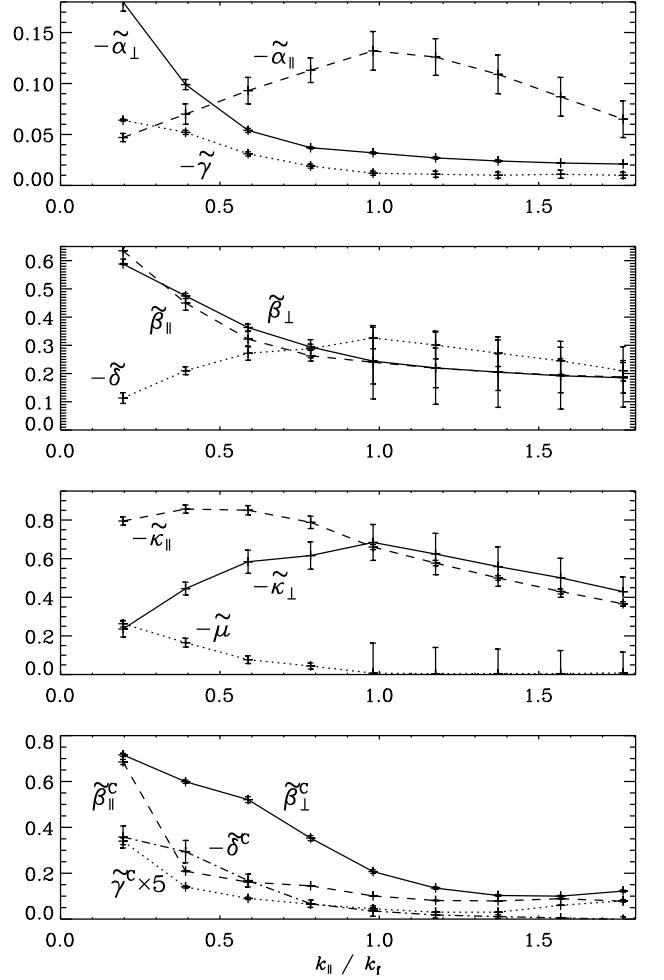


Fig. 6. k_\parallel dependence of transport coefficients in a model with rotation and density stratification, $k_\perp = \sqrt{2}k_1$, $P_m = Sc = 1$, $R_m = 12$, $Co = 1.0$, $Gr = 0.16$, $k_f/k_1 = 5$.

The results are shown in Figure 8. As expected, all quantities increase approximately linearly with R_m if $R_m < 1$, or with P_m if $P_m < 1$, and seem to level off to constant values for larger values of R_m , or P_m , although the uncertainty tends to increase significantly.

6. Conclusions

In this paper we have dealt with the mean electromotive force and the mean passive scalar flux in axisymmetric turbulence and have calculated the transport coefficients that define these quantities. Unlike most of the earlier work, we have no longer assumed that mean fields are defined as planar averages but admit a dependence on all three space coordinates. The number of test fields and test scalars is the same (4 and 2, respectively) as in earlier work using planar averages, so the computational cost is unchanged.

We may conclude from general symmetry considerations that the mean electromotive force $\overline{\mathcal{E}}$ has altogether nine contributions: three defined by the mean magnetic field $\overline{\mathbf{B}}$, three by the mean current density $\overline{\mathbf{J}}$, and three by the vector $\overline{\mathbf{K}}$, which is the projection of the symmetric part of the gradient tensor $\nabla \mathbf{B}$ of the magnetic field on the preferred direction. In many representations of $\overline{\mathcal{E}}$ the last three contri-

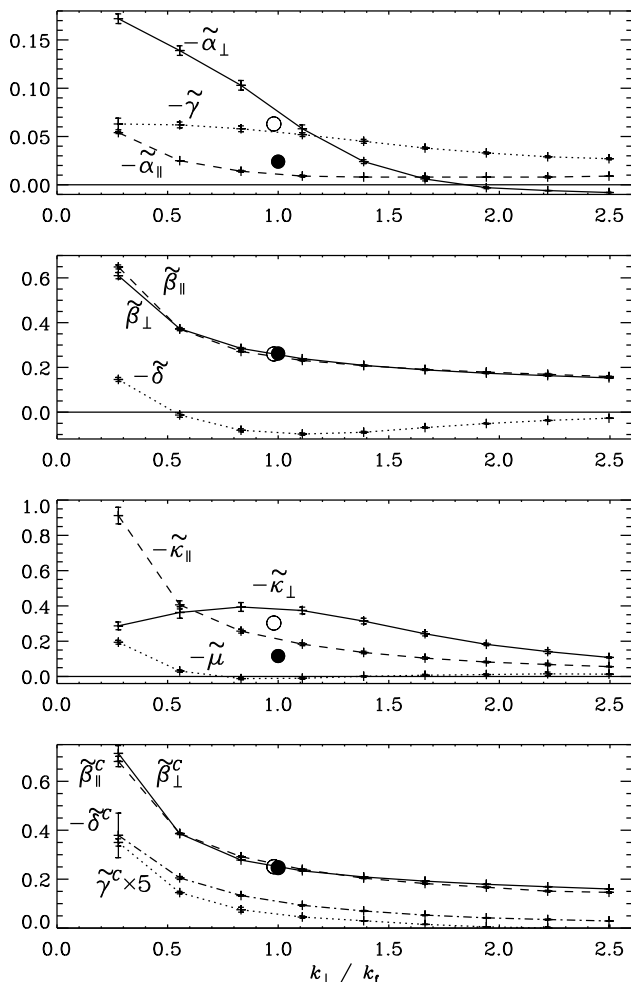


Fig. 7. Same as Figure 6, but k_{\perp} dependence, $k_{\parallel} = k_1$. The filled and open circles denote results for α_{\perp} , β_{\perp} , κ_{\perp} , and β_{\perp}^c obtained with $k_x/k_y = 0.75$ [$\mathbf{k}_{\perp} = (3, 4, 0)k_1$] and $k_x/k_y = 0.2$ [$\mathbf{k}_{\perp} = (1, 5, 0)k_1$], respectively.

butions have been ignored. Our results underline that this simplification is in general not justified. The corresponding coefficients κ_{\perp} , κ_{\parallel} and μ are in general not small compared to β_{\perp} , β_{\parallel} and δ .

It has been known since long that a stratification of the turbulence intensity, that is, a gradient of $\overline{\mathbf{u}^2}$, causes a pumping of magnetic flux (Rädler, 1969a). It remained however uncertain whether the same effect occurs if a preferred direction is given by a gradient of the mean mass density $\overline{\rho}$ while the turbulence intensity is spatially constant. In our calculations, which correspond to this assumption, the value of γ is not clearly different from zero. This suggests that a gradient of the mass density alone is not sufficient for pumping, what is also in agreement with results of Brandenburg et al. (2011). This is even more remarkable as the corresponding coefficient γ^c which describes the transport of a mean passive scalar is noticeably different from zero. Pumping down the density gradient is indeed expected (Elperin et al., 1995). An explanation of these results would be very desirable.

In homogeneous rotating turbulence, apart from an anisotropy of the mean-field conductivity, the $\boldsymbol{\Omega} \times \overline{\mathbf{J}}$ effect occurs (Rädler, 1969b). In the passive scalar case again an

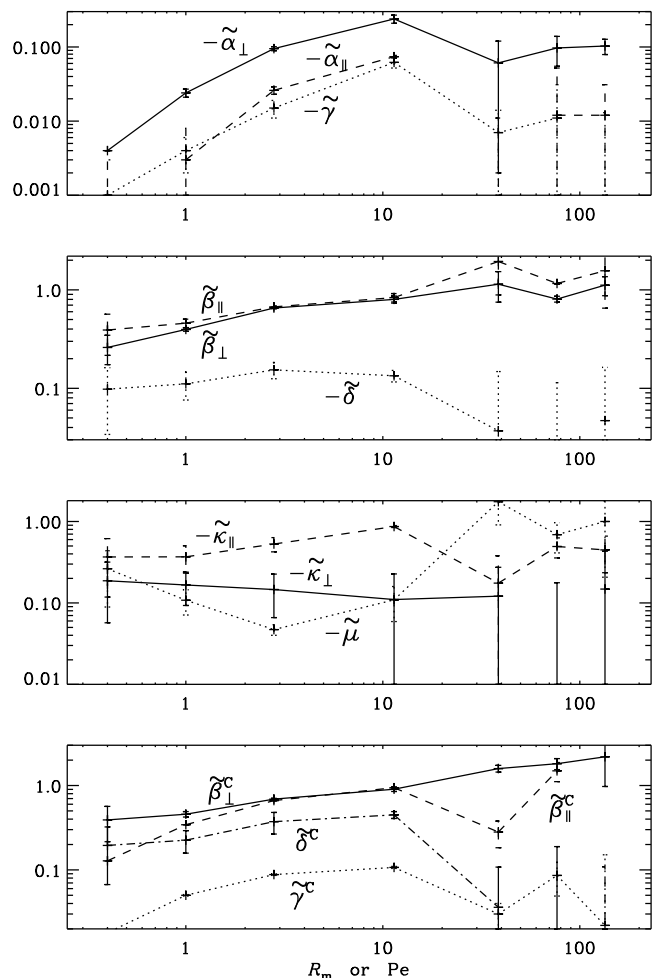


Fig. 8. Dependencies of the transport coefficients on R_m or Pe in a model with rotation and density stratification, $P_m = Sc = 1$, $Co = 1.0$, $Gr = 0.16$, $k_f/k_1 = 5$.

anisotropy of the mean diffusivity is possible. Even if the flux proportional to $\boldsymbol{\Omega} \times \nabla \overline{C}$ is non-zero, it cannot influence \overline{C} .

Let us turn to the induction effects described by $\overline{\mathbf{K}}$. If the preferred direction is given by a polar vector, the corresponding contribution to the mean electromotive force can only be proportional to $\hat{\mathbf{e}} \times \overline{\mathbf{K}}$. We found such a contribution in the case of the Roberts flow and also, for turbulence subject the Coriolis force, in the results presented in Figure 2 and Figures 4–7.

Contributions to the mean electromotive force as described here by $\overline{\mathbf{K}}$ occur also in earlier calculations, e.g. Kitchatinov et al. (1994) or Rüdiger & Brandenburg (1995). As a consequence of other notations, however, this is not always obvious. For example, Rüdiger & Brandenburg (1995) consider a mean electromotive force of the form

$$\overline{\boldsymbol{\mathcal{E}}} = -\eta_{\parallel} \overline{\mathbf{J}} + (\eta_{\parallel} - \eta_T) (\hat{\mathbf{z}} \overline{J}_z - \hat{\mathbf{z}} \times \nabla \overline{B}_z) \quad (30)$$

with two coefficients η_{\parallel} and η_T (equation (18) of their paper with $\mu_0 \overline{\mathbf{J}}$, in the sense of the definition introduced here, replaced by $\overline{\mathbf{J}}$; $\hat{\mathbf{z}}$ is our $\hat{\mathbf{e}}$). It is equivalent to our representations (3) or (4) of $\overline{\boldsymbol{\mathcal{E}}}$ if we put there $\beta_{\perp} = \frac{1}{2}(\eta_{\parallel} + \eta_T)$, $\beta_{\parallel} = \eta_T$, $\mu = \eta_{\parallel} - \eta_T$ and all other coefficients equal to zero. This implies $\beta_{\perp} - \beta_{\parallel} = \mu/2$, which is in agreement

with the relation for μ in equation (28) for the Roberts flow. The latter equality is also approximately obeyed for turbulence in the presence of rotation, stratification, and both; see Figures 2 and 4 and 5, respectively.

If both rotation and density stratification are present, there is also an α effect, which is necessarily anisotropic. For strong rotation $|\alpha_{\parallel}|$ is only half as big as $|\alpha_{\perp}|$. For $\text{Co} \approx 1$, we have $\beta_{\perp} > \beta_{\parallel}$, which means that the magnetic diffusivity along the rotation axis is enhanced. The same is true for the passive scalar diffusivity, where enhanced diffusivity along the rotation axis means $\beta_{\parallel}^C > \beta_{\perp}^C$. Moreover, stratification has the effect of decreasing the magnetic and passive scalar diffusivity in the vertical direction. In the presence of rotation and density stratification all three contributions to the mean electromotive force described by $\overline{\mathbf{K}}$ are in general non-zero. Here, $|\kappa_{\perp}|$ is smaller than $|\kappa_{\parallel}|$.

The present work is applicable to investigations of stellar convection either with or without rotation, and it would provide a more comprehensive description of turbulent transport properties than what has been available so far (Käpylä et al., 2009). The methods utilized in this paper can be extended to a large class of phenomena in which turbulence with just one preferred direction plays an important role. Examples for that include turbulence under the influence of a strong magnetic field and/or an externally applied electric field leading to a current permeating the system. Turbulence generated by the Bell (2004) instability is an example. Other examples include all types of inhomogeneous turbulence with only one preferred direction. In addition to density stratification, there can be a systematic variation of the turbulence intensity in one direction. A further example is entropy inhomogeneity combined with gravity giving rise to Brunt-Väisälä oscillations. Pumping effects also exist in homogeneous flows if the turbulence is helical (Mitra et al., 2009). By contrast, shear problems or other types of problems with two or more preferred directions that are inclined to each other (e.g., turbulence in a local domain of a rotating stratified shell at latitudes different from the two poles) are not amenable to such a study. Of course, although we refer here to axisymmetric turbulence, problems in axisymmetric cylindrical geometry are also not amenable to this method, because the turbulence must be homogeneous in one plane.

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Appendix A: Derivation of relation (3)

We start from the aforementioned assumption according to which \mathcal{E} is linear and homogeneous in $\overline{\mathbf{B}}$ and its first spatial derivatives,

$$\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j + b_{ijk}(\nabla\overline{\mathbf{B}})_{jk}. \quad (\text{A.1})$$

Here a_{ij} and b_{ijk} are tensors determined by the fluid flow. The gradient tensor $(\nabla\overline{\mathbf{B}})_{jk}$ can be split into a antisym-

metric part, which can be expressed by $\overline{\mathbf{J}}$, and a symmetric part $(\nabla\overline{\mathbf{B}})_{jk}^S$. Therefore we may also write

$$\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j - b_{ij}\overline{J}_j - c_{ijk}(\nabla\overline{\mathbf{B}})_{jk}^S \quad (\text{A.2})$$

with new tensors b_{ij} and c_{ijk} , the latter being symmetric in j and k . From the further assumption that the flow constitutes an axisymmetric turbulence we may conclude that a_{ij} , b_{ij} and c_{ijk} are axisymmetric tensors. Defining the preferred direction by the unit vector \hat{e} we then have

$$\begin{aligned} a_{ij} &= a_1\delta_{ij} + a_2\epsilon_{ijl}\hat{e}_l + a_3\hat{e}_i\hat{e}_j, \\ b_{ij} &= b_1\delta_{ij} + b_2\epsilon_{ijl}\hat{e}_l + b_3\hat{e}_i\hat{e}_j, \\ c_{ijk} &= c_1\delta_{jk}\hat{e}_i + c_2(\delta_{ij}\hat{e}_k + \delta_{ik}\hat{e}_j) \\ &\quad + c_3(\epsilon_{ijl}\hat{e}_l\hat{e}_k + \epsilon_{ikl}\hat{e}_l\hat{e}_j) + c_4\hat{e}_i\hat{e}_j\hat{e}_k, \end{aligned} \quad (\text{A.3})$$

with coefficients a_1, a_2, \dots, c_4 determined by the fluid flow. Taking (A.2) and (A.3) together and considering that

$$\begin{aligned} (\delta_{ij}\hat{e}_k + \delta_{ik}\hat{e}_j)(\nabla\overline{\mathbf{B}})_{jk}^S &= 2\overline{K}_i, \\ (\epsilon_{ijl}\hat{e}_l\hat{e}_k + \epsilon_{ikl}\hat{e}_l\hat{e}_j)(\nabla\overline{\mathbf{B}})_{jk}^S &= -2(\hat{e} \times \overline{\mathbf{K}})_i, \\ \hat{e}_i\hat{e}_j\hat{e}_k(\nabla\overline{\mathbf{B}})_{jk}^S &= (\hat{e} \cdot \overline{\mathbf{K}})\hat{e}_i, \end{aligned} \quad (\text{A.4})$$

we find

$$\begin{aligned} \overline{\mathcal{E}} &= a_1\overline{\mathbf{B}} - a_2\hat{e} \times \overline{\mathbf{B}} - a_3(\hat{e} \cdot \overline{\mathbf{B}})\hat{e} \\ &\quad + b_1\overline{\mathbf{J}} - b_2\hat{e} \times \overline{\mathbf{J}} - b_3(\hat{e} \cdot \overline{\mathbf{J}})\hat{e} \\ &\quad + 2c_2\overline{\mathbf{K}} - 2c_3\hat{e} \times \overline{\mathbf{K}} + c_4(\hat{e} \cdot \overline{\mathbf{K}})\hat{e}. \end{aligned} \quad (\text{A.5})$$

Since $(\nabla\overline{\mathbf{B}})_{ii} = 0$ there is no contribution with c_1 . With a proper reaming of the coefficients (A.5) turns into (3).

References

- Bell, A. R. 2004, MNRAS, 353, 550
Brandenburg, A., & Sokoloff, D. 2002, Geophys. Astrophys. Fluid Dyn., 96, 319
Brandenburg, A., & Subramanian, K. 2005, Phys. Rep., 417, 1
Brandenburg, A., Rädler, K.-H., & Schrunner, M. 2008a, A&A, 482, 739
Brandenburg, A., Rädler, K.-H., Rheinhardt, M., & Käpylä, P. J. 2008b, ApJ, 676, 740
Brandenburg, A., Svedin, A., & Vasil, G. M. 2009, MNRAS, 395, 1599
Brandenburg, A., Kemel, K., Kleorin, N., & Rogachevskii, I. 2011, ApJ, submitted
Chatterjee, P., Mitra, D., Rheinhardt, & M. Brandenburg, A. 2011, A&A, 534, A46
Elperin, T., Kleorin, N., Rogachevskii, I. 1995, Phys. Rev. Lett., 52, 2617
Elperin, T., Kleorin, N., Rogachevskii, I. 1996, Phys. Rev. Lett., 76, 224
Haugen, N.E.L., Brandenburg, A., Dobler, W. 2004, Phys. Rev. E, 70, 016308
Hubbard, A., & Brandenburg, A. 2009, ApJ, 706, 712
Käpylä, P. J., Korpi, M. J., & Brandenburg, A. 2009, A&A, 500, 633
Kitchatinov, L. L., Rüdiger, G., Pipin, V. V. & Rüdiger, G. 1994, Astron. Nachr., 315, 157
Krause, F., & Rädler, K.-H., 1971, In R. Rompe and M. Steenbeck, *Ergebnisse der Plasmaphysik und der Gaselektronik* Band 2, Akademie-Verlag Berlin, pp. 6–154
Krause, F., & Rädler, K.-H., 1980, *Mean-Field Magnetohydrodynamics and Dynamo Theory*, Akademie-Verlag Berlin and Pergamon Press Cambridge
Madarassy, E. J. M., & Brandenburg, A. 2010, Phys. Rev. E, 82, 016304
Miesch, M. S., Brandenburg, A., & Zweibel, E. G. 2000, Phys. Rev., E61, 457
Mitra, D., Käpylä, P. J., Tavakol, R., & Brandenburg, A. 2009, A&A, 495, 1

- Prandtl, L. 1925, *Zeitschr. Angewandte Math. Mech.*, 5, 136
- Rädler, K.-H. 1966, Thesis Univ. Jena
- Rädler, K.-H. 1968, *Z.Naturforsch.* 23a, 1851
- Rädler, K.-H. 1969a, *Mber. Dt. Akad. Wiss* 11, 194
- Rädler, K.-H. 1969b, *Geod. Geophys. Veröffentlichungen Reihe II Heft* 13, 131
- Rädler, K.-H. 1976, In V. Bumba and J. Kleczek, *Basic Mechanisms of Solar Activity*, D. Reidel Publishing Company Dordrecht, pp. 323-344
- Rädler, K.-H., Rheinhardt, M., Apstein, E., & Fuchs, H., *Magnetohydrodynamics* 38, 39, 2002a
- Rädler, K.-H., Rheinhardt, M., Apstein, E., & Fuchs, H., *Nonlinear Processes in Geophysics* 9, 171, 2002b
- Rädler, K.-H., Kleorin, N., & Rogachevskii, I. 2003, *Geophys. Astrophys. Fluid Dyn.*, 97, 249
- Rädler, K.-H., Brandenburg, A., Del Sordo, F., & Rheinhardt, M. 2011, *Phys. Rev.*, E84, 046321
- Roberts, P. H., & Soward, A. M. 1975, *Astron. Nachr.*, 296, 49
- Rüdiger, G. & Brandenburg, A. 1995, *A&A*, 296, 557
- Schrinner, M., Rädler, K.-H., Schmitt, D., Rheinhardt, M., Christensen, U. 2005, *Astron. Nachr.*, 326, 245
- Schrinner, M., Rädler, K.-H., Schmitt, D., Rheinhardt, M., Christensen, U. R. 2007, *Geophys. Astrophys. Fluid Dyn.*, 101, 81
- Steenbeck, M., Krause, F., & Rädler, K.-H. 1966, *Z. Naturforsch.*, 21a, 369
- Sur, S., Brandenburg, A., & Subramanian, K. 2008, *MNRAS*, 385, L15
- Vitense, E. 1953, *Z. Astrophys.*, 32, 135