# 1 The $Q$ product

The pseudoscalar of interest here is given by

$$Q = (\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2) \cdot \boldsymbol{n}_3. \quad (1)$$

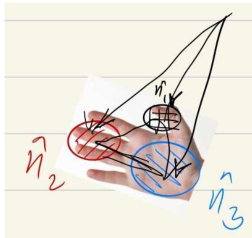

Figure 1: Geometry for right-handed ($Q > 0$).

This scalar triple product of three vectors is a pseudoscalar whose magnitude is given by one quarter of the volume spanned by these three vectors. Here, $\hat{\boldsymbol{n}}_1$, $\hat{\boldsymbol{n}}_2$, and $\boldsymbol{n}_3$ are unit vectors, so their lengths is the same and the volume is approximately given by twice the area $A$ of the triangle in the sky; see Fig. 1. Thus, larger triangles in the sky contribute more to $Q$ than smaller ones.

The signs of this scalar triple product depends on the relative orientation of the three vectors. Therefore, each of the vectors must be uniquely distinguishable from the others. In the work of Tashiro et al. (2014), it was the energy (low, intermediate, and high), which can be put in a unique order. In some of our sketches and illustrations, we order the three vectors by color in the order black, red, and blue.

By summing over all possible triangles in the sky, we get contributions of different signs and magnitude. The net result depends on the final outcome of many cancellations and is dominated by the contributions from the largest triangle, which gives the dominant contribution.

In the following, we want to estimate the net result of these cancelations. We consider only three groups of vectors that are distinguished by the energy of the corresponding photon with the directions $\hat{\boldsymbol{n}}_\alpha^{(i)}$ for $\alpha = 1$, 2, and 3 for the three groups with increasing energy, and the index $i$ distinguishes the different members of the same group. The total number of members of each of the three groups are $N_1$, $N_2$, and $N_3$, respectively. Thus, the average $Q$ for all triangles is

$$\overline{Q} = \frac{1}{N_1 N_2 N_3} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} \left( \hat{\boldsymbol{n}}_1^{(i)} \times \hat{\boldsymbol{n}}_2^{(j)} \right) \cdot \boldsymbol{n}_3^{(k)}, \quad (2)$$

where the overbar denotes averaging. Since the three indices are independent of each other, we can write

$$\overline{Q} = (\overline{\boldsymbol{n}}_1 \times \overline{\boldsymbol{n}}_2) \cdot \overline{\boldsymbol{n}}_3, \quad (3)$$

where

$$\overline{\boldsymbol{n}}_\alpha = \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} \hat{\boldsymbol{n}}_\alpha^{(i)} \quad (4)$$

for $\alpha = 1$, 2, and 3. The end result therefore reduces therefore to the scalar triple product of just these three average vectors. They are no longer unit vectors, and their modulus could instead be zero, if all photons are uniformly distributed over the sky. Any nonuniformity, which can already be caused by a nonuniform distribution of exposure times over the sky, would yield a finite average.

It is convenient to express $\overline{\boldsymbol{n}}_\alpha$ again in terms of a unit vector and a mean amplitude. We can call this unit vector again $\overline{\boldsymbol{n}}_\alpha$, because it plays the same role as the unit vector in Equation (1). It is distinguished from the unit vectors of individual photons, $\hat{\boldsymbol{n}}_\alpha^{(i)}$ by the absence of the upper index. Thus, we can write

$$\overline{\boldsymbol{n}}_\alpha(\hat{\boldsymbol{n}}) = p_\alpha(\hat{\boldsymbol{n}})\, \hat{\boldsymbol{n}}_\alpha \quad (5)$$

Here, the $\hat{\boldsymbol{n}}$ without any indices denotes just the independent coordinate in the sky and is given by

$$\hat{\boldsymbol{n}} = \begin{pmatrix} \sin\theta \, \cos\phi \\ \sin\theta \, \sin\phi \\ \cos\theta \end{pmatrix}, \quad (6)$$

where $\theta$ is colatitude and $\phi$ is longitude.

# 2 Examples

The value of the average $Q$ depends on the radius $\sigma_\alpha$ of each of the three patches. We can model these three patches by a Gaussian profile for the probability density of having a photon in a particular direction $\hat{\boldsymbol{n}}$ in the sky as

$$p_\alpha(\hat{\boldsymbol{n}}) = \frac{p_{\alpha 0}}{(2\pi\sigma^2)^{3/2}} \exp\left[\mathrm{acos}^2(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}_\alpha)/2\sigma_\alpha^2\right] \quad (7)$$

where $p_\alpha$ is the amplitude or probability of having a photon anywhere in the sky. Note that

$$\int p_\alpha(\hat{\boldsymbol{n}})\, \mathrm{d}\Omega_{\hat{\boldsymbol{n}}} = p_{\alpha 0}. \quad (8)$$

Given the enormous amount of cancelation, the crucial question is whether a physically meaningful net effect can exist at all and under what conditions. In particular, we expect such a physically meaningful contribution to be isotropically distributed over the whole sky.

To approach this questions, let us begin with a simple example of many small triangles, all of which have the same relative handedness.

## References

Tashiro, H., Chen, W., Ferrer, F., & Vachaspati, T. 2014, MNRAS, 445, L41

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