Dynamos in ideal magnetohydrodynamics?

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1 Introduction

Ideal magnetohydrodynamics (MHD) describes the magnetic field evolution in a perfectly conducting fluid and is given by the induction equation of the form

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{1}$$

where $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{u}$ is the velocity, and t is time. This equation is routinely stated in the astrophysical literature, especially in the context of numerical simulations of the MHD equations, where some sort of magnetic diffusion and the corresponding dissipation are usually implied by the particular numerical scheme under consideration (Stone & Norman, 1992; Pen et al., 2003; Derigs et al., 2016). In many cases, the corresponding operator cannot easily be expressed in terms of explicit mathematical operations. This is one reason for omitting the diffusion and dissipation terms in the astrophysical literature. Another reason is the fact that the magnetic diffusivity is very small in most astrophysical settings. This does not generally imply that the magnetic dissipation is small (Hendrix et al., 1996; Galsgaard & Nordlund, 1996), although it can be small if the magnetic Prandtl number is large, i.e., the kinematic viscosity is much larger than the magnetic diffusivity (Bra14,BR19).

We now know that magnetic field generation by dynamo action is a generic process in most astrophysical flows, because they tend to be turbulent, so a certain fraction of the kinetic energy will be diverted into magnetic energy (BL13). Thus, the process of dynamo action is a typical ingredient of astrophysical turbulence simulations. This raises the question whether dynamo action of any type is possible at all when Equation (1) is solved and the magnetic diffusivity is thus strictly zero. Early work of Moffatt & Proctor (1985) demonstrated that, in the special case of steady flows, the eigenvalue of Equation (1) must be zero in the strictly ideal case.

The result of Moffatt & Proctor (1985) is somewhat counterintuitive, because one expects the magnetic field always to grow when there is no magnetic diffusivity. What happens, however, at least for a steady flow, is that the magnetic field develops progressively smaller structures, so the magnetic field increases—potentially even exponentially—by concentrating on itself into ever tinier instructions. Because the field structure changes all the time, this field does not correspond to an eigenfunction.

The situation may be different in a turbulent and thus time-dependent flow, where the velocity is constantly changing before the field has a chance to concentrate itself too much. This may lead to a field that has statistically always the same typical size. Whether or not this really happens is unclear and needs to be investigated. Studying this in more detail is the main purpose of the present work.

2 Models

2.1 Analysis tools

A and EP methods, resetting $\boldsymbol{B}$ to $\boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta$, compare $B_{\rm rms}^{\rm EP}/B_{\rm rms}^{\rm A}$ versus time, compare $J_{\rm rms}/B_{\rm rms}$ versus time (smaller structures), kurtosis and pdfs of B_i, magnetic helicity spectra $H_{\rm M}(k,t)$.

The present experiment also allows us to address the question what happens with magnetic helicity. Even when writing the magnetic vector potential in the more general form as

$$\boldsymbol{A} = q\alpha\boldsymbol{\nabla}\beta - (1-q)\beta\boldsymbol{\nabla}\alpha, \tag{2}$$

where $0 \le q \le 1$ is some weighting factor, the magnetic field is still always independent of q, i.e., the local magnetic helicity density is always zero. On the other hand, it is still possible to have a non-vanishing magnetic helicity spectrum, which is defined as (cf. **?**)

$$H_{\rm M}(k) \quad = \quad \tfrac{1}{2}\sum_{k_-<|\boldsymbol{k}|\le k_+} (\tilde{\boldsymbol{A}}\cdot\tilde{\boldsymbol{B}}^* + \tilde{\boldsymbol{A}}^*\cdot\tilde{\boldsymbol{B}}), \tag{3}$$

where $k_\pm = k \pm \delta k/2$ and $\delta k = 2\pi/L$ is the wavenumber increment and also the smallest wavenumber in the domain L^3.

2.2 ABC flows

To understand the nature of dynamo action in the steady ABC flow (with $A = B = C = 1$), we compare solutions with the A and EP methods at progressively smaller diffusivity.

The initial magnetic field is expressed in terms of Euler potentials and is given by $\alpha = \cos y$ and $\beta = \cos z$. Figure 1 shows that there is an initial windup phase during which the magnetic field increases approximately exponentially or even super-exponentially. Later, a truly exponential growth commences, although the growth is then less rapid than during the windup phase.

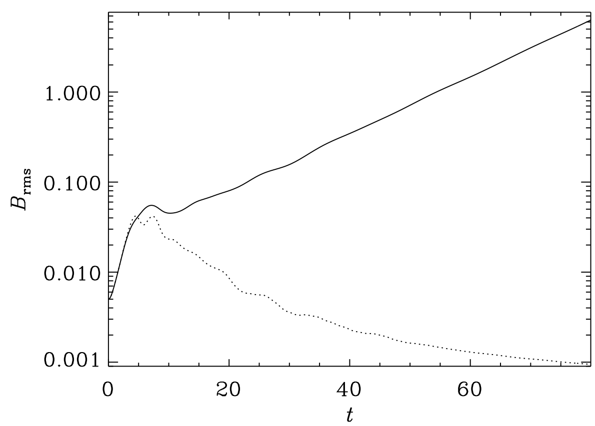

Figure 1: Dynamo action for $\eta = 10^{-3}$ at 256^3 with the A method (solid line) and magnetic field decay with the EP method (dotted line).

Table 1:

Run	N	η	$k_{\rm Tay}^{A}$	$k_{\rm Tay}^{EP}$
A	256	10^{-3}	20.8	20.2
B	256	5×10^{-4}	29.8	28.5
C	512	2×10^{-4}	46.0	45.7
D	1024	10^{-4}	65.9	67.1
E	1024	5×10^{-5}	95.0	103.6
F	1024	2×10^{-5}	158.6	262.4

Figure 2 shows that the windup phase is prolonged as one decreases the magnetic diffusivity. This phase can also be described with the EP method at finite magnetic diffusivity.

Figure 4 shows that at $t = 10$, the A and EP methods agree reasonably well, although both methods suffer from poor resolution at the lowest magnetic diffusivity.

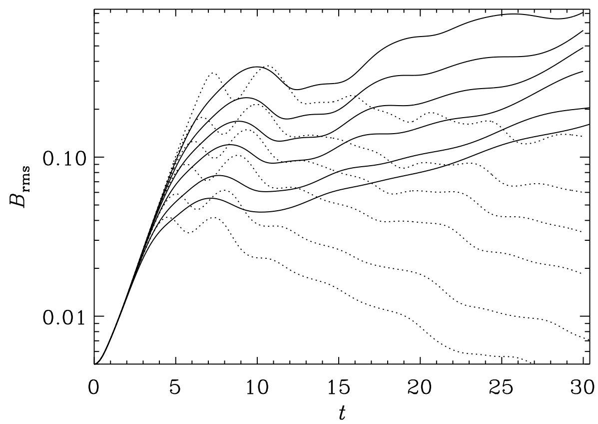

Figure 2: Similar to Fig. 1 but for $\eta = 10^{-3}$ (lowermost lines), 5×10^{-4} (both at 256^3), 2×10^{-4} (at 512^3), 10^{-4}, 5×10^{-5}, and 2×10^{-5} (at 1024^3). Note that the EP method agrees with the A method only during the initial wind-up phase, and never during the later dynamo phase.

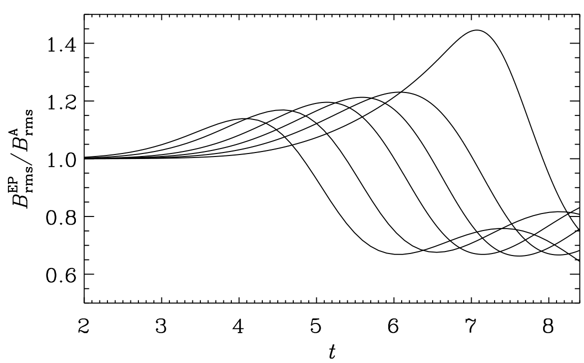

Figure 3: The ratio $B_{\rm rms}^{\rm EP}/B_{\rm rms}^{\rm A}$ during early times.

At a given resolution, here 1024^3, the EP method reproduces the initial windup phase until a certain time. At higher resolution, the time over which the EP method agrees with the A method will be longer. In the early phase, however, the magnetic field evolution appears to be similar to the windup in 2-D, when there is no dynamo.

Figure 5 shows the pdf of the 3 components of $\boldsymbol{B}$ at $t = 7$ with the EP and A methods, respectively, and compares in a log-log plot. With the EP method, the pdf is more extended, but the magnetic field decays and at later times, only the pdf obtained with the A method has extended tails; see Fig. 6 for $t = 30$. In all cases, the PDF has power-law tails proportional to $1/B_i^2$.

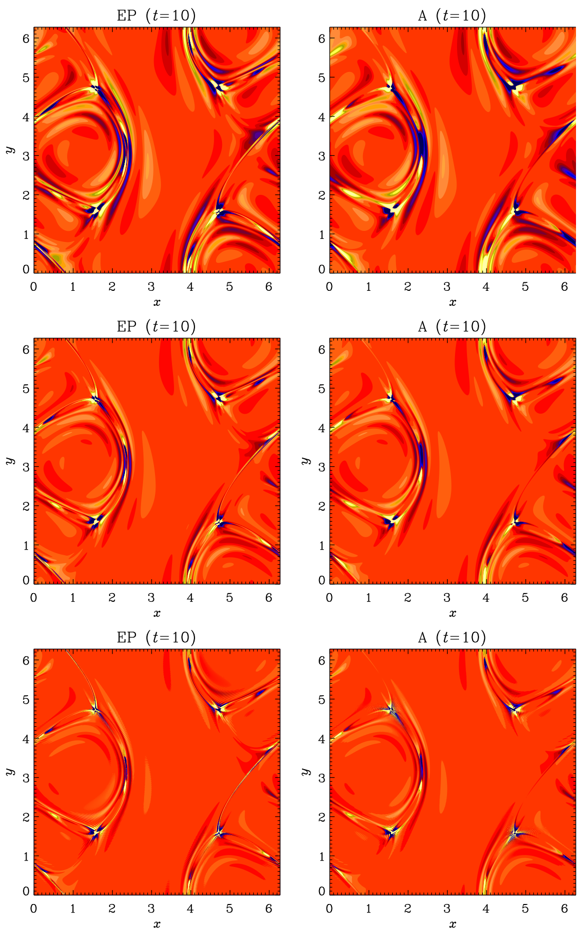

Figure 4: $B_z(x, y)$ in a given plane z = const for η =1e-4, 5e-5, and 2e-5. In all cases, the resolution is 1024^3.

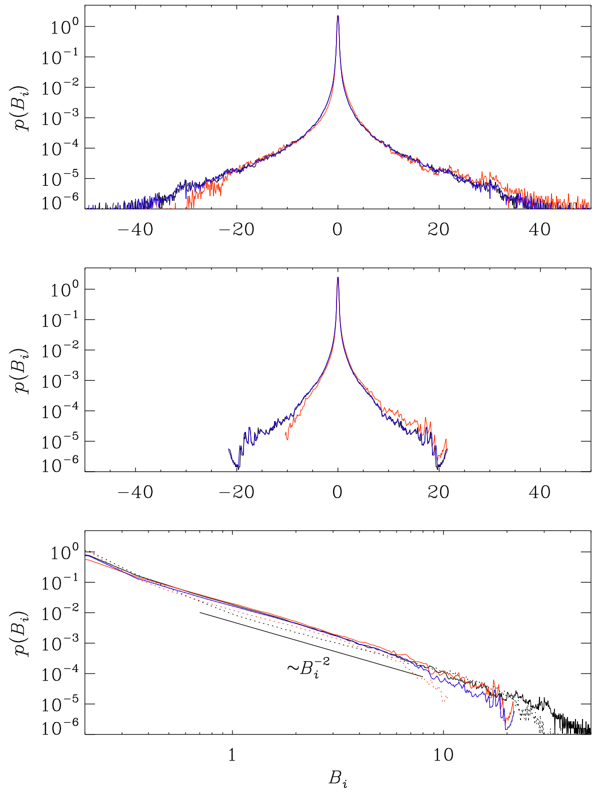

Figure 5: Histograms for the ABC flow at $t = 7$ for the run with $\eta = 2 \times 10^{-5}$.

3 Galloway–Proctor flow

The Galloway–Proctor flow is time dependent. We use here the circular polarized version in the definition of GP92. Figure 7 shows B_x for $t = 10$ (title says *incorrectly* $t = 1$), and $t = 60$ (second row) for $\eta = 10^{-3}$, and then the same for $\eta = 10^{-4}$ and $\eta = 10^{-5}$ in rows 3 and 4. Figure 8 shows the time evolution for all 3 cases. The EP solution departs from the correct solution for $t \gtrsim 4$. The A solution with $\eta = 10^{-5}$ is under-resolved.

4 Delta-correlated turbulence

In mean-field electrodynamics (Krause & Rädler, 1980), it is shown that there is an α effect even in the absence of magnetic diffusion. This is done by using the high connectivity limit, in which the magnetic diffusion operator is neglected. This approximation is only valid when the Strouhal number $\mathrm{St} = u_{\mathrm{rms}} k_{\mathrm{f}} \tau$ is small, i.e., when the correlation time τ of the flow is zero. To realize such a flow, we now XX

Instead of using a delta-correlated forcing function with plane waves in the momentum equation, as one usually does, we now use this function for the velocity. Figure 9 shows the time evolution. Figure 10 compares $B_x(y, z)$ at $x = 0$ and $t = 200$. Figure 11 compares magnetic energy and helicity spectra for four times. In Figure 11, we compare magnetic energy and helicity spectra from the EP and A methods at four times. We see that $H_{\mathrm{M}}(k, t)$ shows positive and negative contributions of small and large wavenumbers, respectively. This agrees with early studies on the properties of α^2 dynamos (Seehafer, 1996; **?**) and was also seen in the early phase of resistive turbulent α^2 dynamos (Brandenburg, 2001). Interestingly, both the EP and the A methods reproduce this behavior correctly, and even at late times, the EP method recovers this behavior qualitatively.

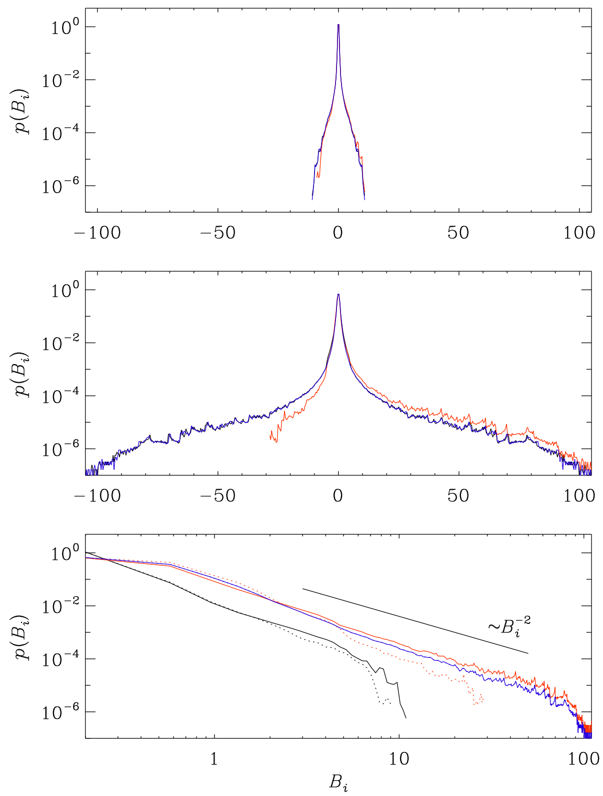

Figure 6: Histograms for the ABC flow at $t = 30$ for the run with $\eta = 2 \times 10^{-5}$.

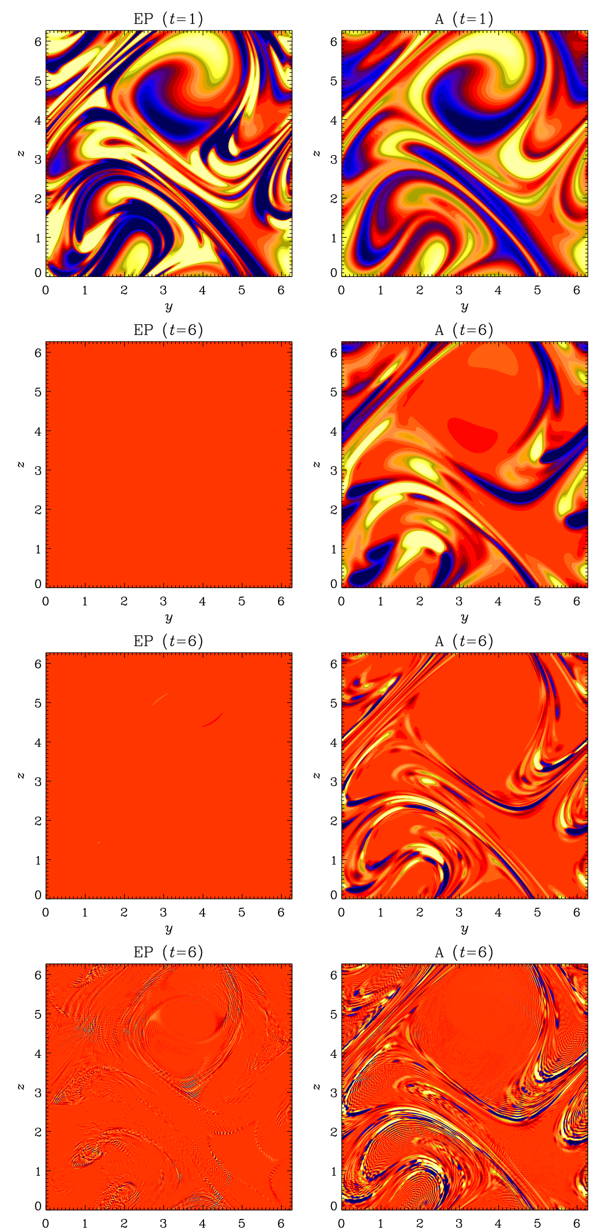

Figure 7: $B_x(y, z)$ at $x = 0$ for η =1e-3, 1e-4, and 1e-5. In all cases, the resolution is 512^3.

References

Brandenburg, A. 2001, ApJ, 550, 824

Derigs, D., Winters, A. R., Gassner, G. J., & Walch, S. 2016, JCoPh, 317, 223

Galsgaard, K., & Nordlund, Å. 1996, J. Geophys. Res., 101, 13445

Hendrix, D. L., van Hoven, G., Mikic, Z., & Schnack, D. D. 1996, ApJ, 470, 1192

Ji, H. 1999, PhRvL, 83, 3198

Krause, F., & Rädler, K.-H. 1980, Mean-field Magnetohydrodynamics and Dynamo Theory (Oxford: Pergamon Press)

Moffatt, H. K., & Proctor, M. R. E. 1985, J. Fluid Mech., 154, 493

Pen, U.-L., Arras, P., & Wong, S. 2003, ApJS, 149, 447

Seehafer, N. 1996, Phys. Rev. E, 53, 1283

Stone, J. M., & Norman, M. 1992b, ApJS, 80, 791

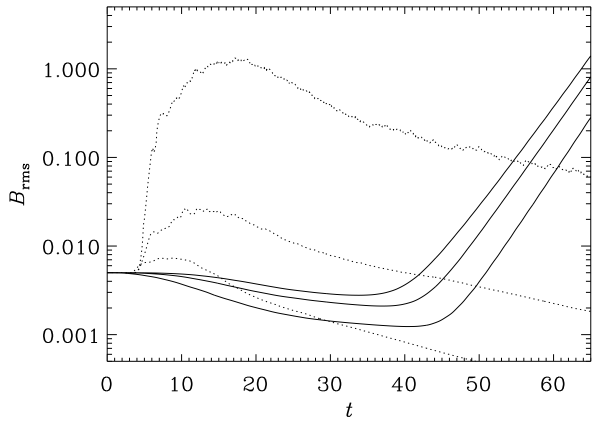

Figure 8: Time evolution for GP flow.

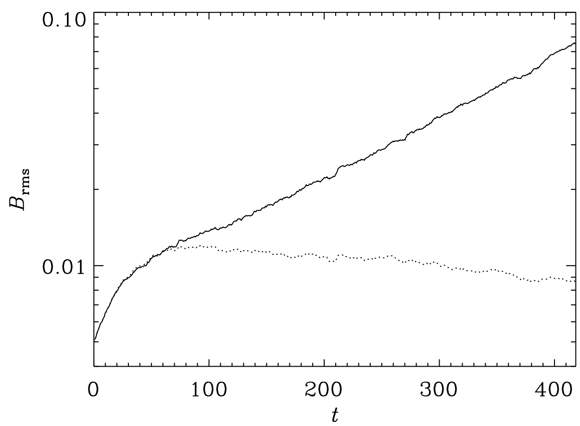

Figure 9: Time evolution for a delta-correlated fully helical flow.

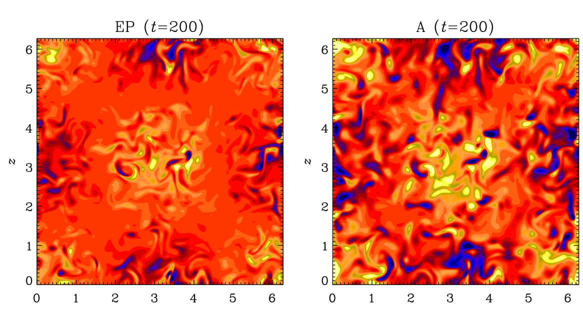

Figure 10: $B_x(y, z)$ at $x = 0$ for the delta-correlated fully helical flow.

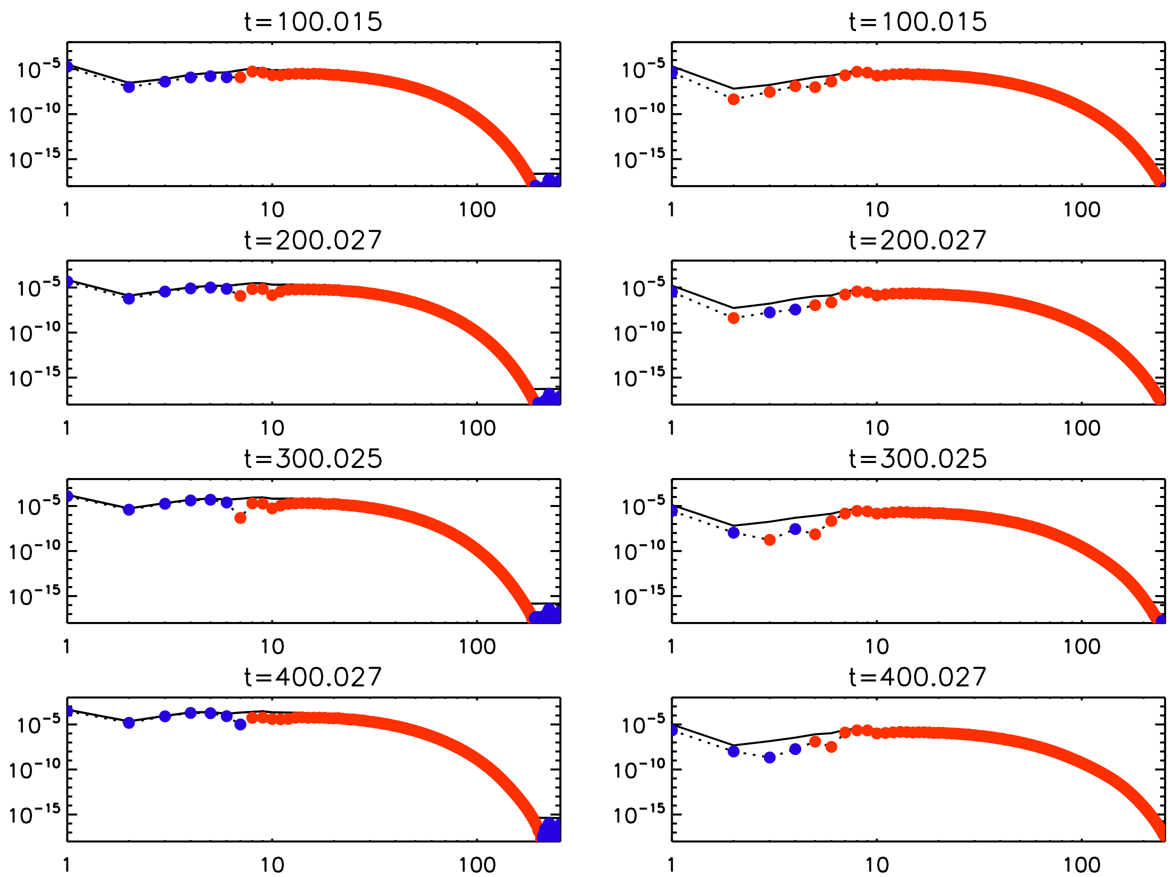

Figure 11: Magnetic energy and helicity spectra for the delta-correlated fully helical flow.