MAGNETIC PRANDTL NUMBER DEPENDENCE OF KINETIC TO MAGNETIC DISSIPATION RATIO

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ABSTRACT

Using direct numerical simulations of three-dimensional hydromagnetic turbulence, either with helical or nonhelical forcing, we show that the ratio of kinetic to magnetic energy dissipation always increases with magnetic Prandtl number, i.e., the ratio of kinematic viscosity to magnetic diffusivity. This dependence can always be approximated by a power law, but the exponent is not the same in all cases. For nonhelical turbulence at large magnetic Prandtl numbers, the exponent is around 1/3, while for all other cases it is between 0.6 and 2/3. In the statistically steady state, the rate of energy conversion from kinetic into magnetic by the dynamo must be equal to the Joule dissipation rate. We emphasize that for both small-scale and large-scale dynamos, the efficiency of energy conversion depends sensitively on the magnetic Prandtl number and thus on the microphysical dissipation process. To understand this behavior, we also study shell models of turbulence and one-dimensional passive and active scalar models. We conclude that the magnetic Prandtl number dependence is qualitatively best reproduced in the one-dimensional model as a result of dissipation via localized Alfvén kinks.

Subject headings: accretion disks — hydrodynamics — MHD — shock waves — turbulence

1. INTRODUCTION

One of the central paradigms of hydrodynamic turbulence is the equivalence of large-scale energy injection and small-scale dissipation into heat through viscosity—regardless how small its value. This is believed to apply also under conditions of astrophysically large Reynolds numbers, when the microphysical viscosity becomes very small compared with the product of the physical scales and velocities of the system. Dramatic examples are quasars, whose luminosity is that of a hundred galaxies and this emission is caused just by the dissipation of turbulence, even though the microphysical viscosity is extremely small. The detailed physical processes are not well understood, but it is now generally believed that they involve magnetic fields as well (Shakura & Sunyaev 1973; Balbus & Hawley 1998).

Indeed, magnetic fields provide an additional important pathway for dissipating turbulent energy through Joule heating. Viscous and Joule dissipation have in common that their heating rates are proportional to the microphysical values of viscosity $\nu$ and magnetic diffusivity $\eta$, respectively. The ratio of these coefficients is the magnetic Prandtl number, $Pr_M = \nu/\eta$. As these coefficients are being decreased, velocity and magnetic field gradients sharpen just enough so that the heating rates remain independent of these coefficients. For the magnetic case of Joule heating, the independence of the magnetic Reynolds number was demonstrated by Galsgaard & Nordlund (1996) and Hendrix et al. (1996) in connection with the coronal heating problem. Over a range of magnetic Reynolds numbers, the approximate constancy of Joule dissipation has also been seen in turbulent dynamo simulations (Candelaresi et al. 2011).

While this picture is appealing and seemingly well confirmed, at least in special cases such as fixed values of $Pr_M$, questions have arisen in cases when the magnetic and fluid Reynolds numbers are changed in such a way that their ratio changes. Hydromagnetic turbulence simulations exhibiting dynamo action have shown that the values of energy dissipation are then no longer constant, and that their ratio scales with $Pr_M$ (Mininni 2007; Brandenburg 2009, 2011a,b). Given that all the energy that is eventually dissipated comes from the forcing in the momentum equation, a change in the dissipation ratio can only be a consequence of a change in the conversion of kinetic to magnetic energy through the dynamo process. Therefore, the dynamo process would intimately be linked to Joule dissipation and one must therefore be concerned that it is also linked to the physical or even numerical nature of energy dissipation. This would be surprising, because dynamo action has frequently been modelled in many astrophysical turbulence simulations by focussing on the so-called ideal equations with just numerical dissipation where no $Pr_M$ can be defined. Examples in the context of local accretion disk dynamo simulations include the papers by Brandenburg et al. (1995), Hawley et al. (1996), and Stone et al. (1996). This leads to an ignorance that is potentially dangerous if such simulations are employed to making predictions about the energy deposition in accretion disks (see discussion by Bisnovaty-Kogan & Lovelace 1997).

There is the concern that the numerical results of Brandenburg (2009, 2011a) are not yet in the asymptotic regime and that the $Pr_M$ dependence might disappear at sufficiently large values of $Re$. However, two arguments against this have now emerged. First, there are analytic results in two-dimensional magnetohydrodynamics (MHD) by Tran et al. (2013) that demonstrate boundedness of mean-squared current density and mean-squared vorticity in the limits of large and small values of $Pr_M$, respectively. In fact, Tran et al. (2013) also produce numerical scalings similar to the results of Brandenburg (2011a,b). Second, MHD shell models of turbulence by Plunian & Stepanov (2010) show for $Pr_M > 1$ a similar $Pr_M$ dependence, which is remarkable because those models can be extended to much larger values of $Re_M$ than what is possibly with DNS.

Thus, there is now mounting evidence for a genuine dependence of macroscopic properties of MHD turbulence on $Pr_M$. Another such dependence has been discussed for some time in connection with nonhelical turbulence exhibiting small-scale
dynamo action in the kinematic regime. Note, however, that this no longer applies in the non-kinematic regime Brandenburg (2011a). For a kinematic small-scale dynamo dynamo, the magnetic energy spectra grow in an approximately shape-
invarent fashion with an approximate $k^{3/2}$ spectrum at small wavenumbers. This spectrum was first predicted by Kazantsev (1968) in the case of a smooth flow. This case corre-
sponds to an idealized representation of turbulence at large
values of $Pr_M$ (Schekochihin et al. 2002), but this spectrum is
apparently also found at small values of $Pr_M$ near unity (see
Fig. 4 of Haugen et al. 2004). Depending on the value of $Pr_M$, the magnetic energy spectrum peaks at wavenum-
bers either within the inertial range of the turbulence or in
the viscous subrange. This has implications for the critical
magnetic Reynolds number for the onset of dynamo action
(Rogachevski & Kleeroin 1997). As explained by Boldyrev &
Cattaneo (2004), the velocity field is rough in the inertial
range. This interpretation has been successfully applied in
clarifying the reason for an apparent divergence (Schekochi-
hin et al. 2005) of the critical Reynolds above which dynamo
action is possible (Iskakov et al. 2007; Schekochihin et al.
2007).

There has been a similar debate regarding the onset of
the magneto-rotational instability in local simulations of acc-
retion disks (Fromang & Papaloizou 2007; Fromang et al.
2007), where the instability was found to be not excited for small values of $Pr_M$. However, these examples are restricted to the physics of small-scale magnetic fields only. If one al-
lowes large-scale fields to develop, e.g., by relaxing the restric-
tion to closed or periodic boundary conditions, this $Pr_M$ de-
pendence disappears (Käpylä & Korpi 2011).

In the following, we shall be concerned with the fully dy-
namic case where kinetic and magnetic energies are com-
parable. The purpose of the present paper is to illuminate the
problem of the $Pr_M$ dependence of the dissipation ratio
through a combination of different approaches to MHD tur-
bulence ranging from direct numerical simulations (DNS) of
the MHD equations in three dimensions and shell models of tur-
bulence capturing aspects of the spectral cascade, to a simple
one-dimensional model of MHD (cf. Thomas 1968; Pouquet
1993; Basu et al. 2014). This leads us to the suggestion that the
$Pr_M$ dependence found in turbulent dynamo simulations is
caused by a dominant influence of dissipative structures on
the turbulent cascade at larger scales. These dissipative struc-
tures can be thought of as local Alfvén kinks whose width is
determined by the algebraic mean of kinematic viscosity and
magnetic diffusivity.

2. SIMULATIONS OF TURBULENT DYNAMOS

2.1. Governing equations

In this section we consider forced MHD turbulence of a gas
that can be described by an isothermal equation of state, i.e.,
the gas pressure $p$ is proportional to the gas density $\rho$ with
$p = \rho c_s^2$, where $c_s = \text{const}$ is the isothermal sound speed.
We apply a forcing function $f$ that is either fully helical or
nonhelical. In both cases, there is initially just a weak seed
magnetic field, which is then amplified by dynamo action. In
the former case with helicity, we obtain large-scale magnetic
fields, as was studied previously with similar setups (Branden-
burg 2001, 2009; Mininni 2007), while in the latter case only
small-scale dynamo action is possible (Cho & Vishniac 2000;
Haugen et al. 2003, 2004; Schekochihin et al. 2004; Branden-
burg 2011a). In some cases we also include the Coriolis force
to study the effects of rotation. We solve the governing equa-
tions in the form

$$\frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{u}, \quad (1)$$

$$\frac{D \mathbf{u}}{Dt} = -\nabla \ln \rho - 2\Omega \times \mathbf{u} + f + \rho^{-1} [J \times \mathbf{B} + \nabla \cdot (2\nu S)], \quad (2)$$

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{J}, \quad (3)$$

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the advective derivative, $\mathbf{u}$ is the velocity, $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field, $A$ is the
magnetic vector potential, $J = \nabla \times \mathbf{B}/\mu_0$ is the current
field, $\mu_0$ is the vacuum permeability, and

$$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{2}\delta_{ij} \nabla \cdot \mathbf{u}$$

is the traceless rate of strain tensor. It is useful to note that

$$\rho^{-1} \nabla \cdot (2\rho S) = \frac{1}{2} \nabla \nabla \mathbf{u} - \nabla \times \nabla \times \mathbf{u} + \mathbf{S} \cdot \nabla \ln \rho, \quad (5)$$

where we call attention to the presence of the $4/3$ factor that will
be relevant for irrotational flows.

We consider a triply periodic domain, so the kinetic and mag-
netic energy balance is described by

$$\frac{d}{dt} \langle p u^2/2 \rangle = \langle p \nabla \cdot \mathbf{u} \rangle + \langle \mathbf{u} \cdot (J \times B) \rangle + \langle \eta \mu_0 \mathbf{J}^2 \rangle, \quad (6)$$

$$\frac{d}{dt} \langle B^2/2\mu_0 \rangle = -\langle \mathbf{u} \cdot (J \times B) \rangle - \langle \eta \mu_0 \mathbf{J}^2 \rangle, \quad (7)$$

where $\mathbf{S} = S_{ij} \delta_{ij}$. The total (kinetic plus magnetic) energy is
sourced by $\langle p \mathbf{u} \cdot \mathbf{f} \rangle$ and dissipated by the sum of viscous
and Joule dissipation, $\epsilon_T = \epsilon_K + \epsilon_M$, with

$$\epsilon_K = \langle 2\rho S^2 \rangle \quad \text{and} \quad \epsilon_M = \langle \eta \mu_0 \mathbf{J}^2 \rangle. \quad (8)$$

The terms $\langle p \nabla \cdot \mathbf{u} \rangle$ and $\langle \mathbf{u} \cdot (J \times B) \rangle$ characterize the work
done by gas expansion and Lorentz force, respectively.

A sketch showing the transfers in and out of the two en-
ergy reservoirs, $E_K = \langle p u^2/2 \rangle$ and $E_M = \langle B^2/2\mu_0 \rangle$, is
given in Figure 1. From this it is clear that, in the steady state,
the quantity $-\langle \mathbf{u} \cdot (J \times B) \rangle$ must be positive and equal to
$\langle \eta \mu_0 \mathbf{J}^2 \rangle$.

2.2. The model

We solve Equations (1)–(3) with periodic boundary condi-
tions using the PENCIL Code\(^1\), which employs sixth or-
der finite differences and a third order accurate time stepping
scheme. For most of our runs we choose a resolution of $512^3$ meshpoints.

In all cases, the amplitude of the forcing function is $f_0 = 0.02$, which results in a Mach number $u_{\text{rms}}/c_s$ of around 0.1. Here, $u_{\text{rms}}$ is the mean root-mean-square (rms) value of the
resulting velocity. The simulations are further characterized by
the fluid and magnetic Reynolds numbers,

$$Re = u_{\text{rms}}/\nu k_l, \quad Re_M = u_{\text{rms}}/\eta k_l,$$

so $Pr_M = Re_M/Re$. In cases with rotation, we also specify the Coriolis number,

$$Co = 2\Omega/u_{\text{rms}} k_l.$$  

\(^1\) http://pencil-code.googlecode.com/
The energy supply for a helically driven dynamo is provided by the forcing function $f = f(x, t)$, which is random in time and defined as

$$f(x, t) = \text{Re}\{N f_{k(t)} \exp[\text{i}(k \cdot x + \sigma \phi(t))]\},$$

where $x$ is the position vector. The wavevector $k(t)$ and the random phase $-\pi < \phi(t) \leq \pi$ change at every time step, so $f(x, t)$ is $\delta$-correlated in time. Therefore, the normalization factor $N$ has to be proportional to $\delta t^{-1/2}$, where $\delta t$ is the length of the time step. On dimensional grounds we choose $N = f_0 c_s (|k| c_s / \delta t)^{1/2}$, where $f_0$ is a nondimensional forcing amplitude. We use $f_0 = 0.02$, which results in a maximum Mach number of about 0.3 and an rms value of about 0.085. At each timestep we select randomly one of many possible wavevectors in a certain range around a given forcing wavenumber with average value $k_i$. Transverse helical waves are produced via (Brandenburg & Subramanian 2005)

$$f_k = R \cdot f_{k(\text{nolinear})} \quad \text{with} \quad R_{ij} = \frac{\delta_{ij} - i\sigma \epsilon_{ijk} k_i}{\sqrt{1 + \sigma^2}},$$

where $\sigma$ is a measure of the helicity of the forcing and $\sigma = 1$ for positive maximum helicity of the forcing function and

$$f_{k(\text{nolinear})} = (k \times \hat{e}) / \sqrt{k^2 - (k \cdot \hat{e})^2}$$

is a nonhelical forcing function, where $\hat{e}$ is an arbitrary unit vector not aligned with $k_i$; note that $|f_k|^2 = 1$ and

$$f_k \cdot (ik \times f_k)^* = 2\sigma k / (1 + \sigma^2),$$

so the relative helicity of the forcing function in real space is $2\sigma / (1 + \sigma^2)$; see Candelaresi & Brandenburg (2013). In the cases mentioned below, we choose $k_i / k_1 = 3.1$ when $\sigma = 1$, so as to allow sufficient scale separation for the large-scale field to develop, and $k_i / k_1 = 1.5$ when $\sigma = 0$, where the issue of scale separation is presumably less critical.

### 2.3. Results

In Table 1 we present a summary of all the runs discussed in this paper. Note first of all that in all cases the energy ratio $E_K / E_M$ is roughly independent of $Pr_M$, but it varies with

![Figure 1](image1.png)

**FIG. 1.** Sketch showing the flow of energy injected by the forcing $(\rho u \cdot f)$ and eventually dissipated viscously and resistively via the terms $\epsilon_K$ and $\epsilon_M$. Note that in the steady state, $\epsilon_M$ must be balanced by $-(u \cdot (j \times B))$.

![Figure 2](image2.png)

**FIG. 2.** Dependence of the ratio $E_K / E_M$ on $Pr_M$ for large-scale (LS) dynamos (solid blue line, Runs A1–C3) and small-scale (SS) dynamos (dashed orange and red lines, Runs X1–Y7).

![Figure 3](image3.png)

**FIG. 3.** Dependence of the dissipation ratio $\epsilon_K / \epsilon_M$ on $Pr_M$ for large-scale dynamos (solid blue line) and small-scale dynamos (dashed orange and red lines). The red filled symbols and black plus signs correspond to the results of Sahoo et al. (2011) for forced and decaying turbulence, respectively, referred to as SPP11 in the legend.

$Re_M$, as was demonstrated previously for the small-scale dynamo (Haugen et al. 2003). For large-scale dynamos, the ratio $E_K / E_M$ is essentially equal to $k_1 / k_1$ (Brandenburg 2001), which is around 0.3 in the present case; see Figure 2. In Figure 3 we show the $Pr_M$ dependence of $\epsilon_K / \epsilon_M$ for $\sigma = 1$ and 0. The simulations show that for both $\sigma = 1$ and 0, the ratio $\epsilon_K / \epsilon_M$ scales with $Pr_M$.

$$\epsilon_K / \epsilon_M \propto Pr_M^q, \quad (15)$$

but the exponent is not always the same. For $\sigma = 1$, we find $q \approx 2/3$ for both small and large values of $Pr_M$, while for $\sigma = 0$, we find $q \approx 0.6$ for $Pr_M < 1$ with $Re \approx 80$ and $q \approx 0.3$ for $Pr_M > 1$ with $Re \approx 460$. For large-scale dynamos ($\sigma = 1$), a similar scaling was first found for $Pr_M \leq 1$ (Mininni 2007; Brandenburg 2009), and later also for $Pr_M \geq 1$ (Brandenburg 2011b). For $Pr_M \leq 1$, this scaling was also found for small-scale dynamos (Brandenburg 2011a), but now we see that for $Pr_M \geq 1$ the slope is smaller.

Our results for $Pr_M > 1$ are compatible with those of Sa-
null
crease of $\epsilon_K/\epsilon_M$. This behavior was partially explained by the findings of Brandenburg (2009, 2011a) that for small values of $\text{Pr}_M = \nu/\eta$, i.e., for $\eta \gg \nu$, most of the spectral energy is dissipated through the magnetic channel, leaving therefore only a reduced amount of kinetic energy to be dissipated, so velocity gradients are not as sharp as in the hydrodynamic case, $\epsilon_K$ is reduced, and so $\epsilon_K/\epsilon_M$ decreases with decreasing values of $\text{Pr}_M$.

Before closing the discussion on the $\text{Pr}_M$ dependence in three-dimensional MHD turbulence, let us make here a comment regarding the work term due to fluid expansion. In all the cases discussed here, $\langle \rho \nabla \cdot \mathbf{u} \rangle$ turns out to be strongly fluctuating, although its time average is a very small fraction of the total energy (0.02%) for our low Mach number runs (Mach numbers around 0.1). There are indications, however, that $\langle \rho \nabla \cdot \mathbf{u} \rangle$ is negative for $\text{Pr}_M < 1$ and positive for $\text{Pr}_M > 1$.

Given that there is currently no phenomenological explanation for the scaling of $\epsilon_K/\epsilon_M$ given by Equation (15), we must consider the possibility that this scaling behavior is not generic and that different scalings can be found in different situations. To shed more light on the possible mechanisms that can explain these scalings, we consider next the results of an MHD shell model of turbulence.

3. SHELL MODELS

Shell models represent the dynamics of turbulence using scalar variables for velocity and magnetic field along logarithmically spaced wavenumbers. The governing equations resemble the original ones with diffusion and forcing terms, as well as quadratic nonlinearities that conserve the same invariants as the original equations: total energy, cross helicity, and a proxy of magnetic helicity. For a recent review of such models, see Plunian et al. (2013). The resulting set of equations can be written as

$$\frac{\partial u_n}{\partial t} = ik_n \left[ N_n(\mathbf{u}, \mathbf{u}) - N_n(\mathbf{b}, \mathbf{b}) \right] - \nu k_n^2 u_n + F_n; \quad (16)$$
$$\frac{\partial b_n}{\partial t} = ik_n \left[ M_n(\mathbf{u}, \mathbf{b}) - M_n(\mathbf{b}, \mathbf{u}) \right] - \eta k_n^2 b_n, \quad (17)$$

where $\mathbf{u} = (u_1, u_2, ..., u_N)$ and $\mathbf{b} = (b_1, b_2, ..., b_N)$ are time-dependent complex vectors representing the state of the system at wavenumbers $k_n = 2^n$ with $n = 1, 2, ..., N$. The nonlinearities are given by (Brandenburg et al. 1996; Frick & Sokoloff 1998)

$$N_n(\mathbf{x}, \mathbf{y}) = x_{n+1}^* y_{n+2}^* - \frac{1}{8} x_{n-1}^* y_{n+1}^* - \frac{1}{8} x_{n-2}^* y_{n-1}^*; \quad (18)$$
$$M_n(\mathbf{x}, \mathbf{y}) = \frac{1}{8} (x_{n+1}^* y_{n+2}^* - x_{n-1}^* y_{n+1}^* - x_{n-2}^* y_{n-1}^*). \quad (19)$$

These equations preserve total energy, cross helicity, and a proxy of magnetic helicity. The only difference between Brandenburg et al. (1996) and Frick & Sokoloff (1998) is a 12/5 scaling factor in front of both nonlinear terms. Models with these coefficients have been used to study the possibility of an inverse cascade in the early universe (Brandenburg et al. 1996, 1997) and the onset properties of small-scale dynamos (Frick & Sokoloff 1998), as well as the possibility of growing dynamo modes from the velocity field of a saturated nonlinear dynamo (Cattaneo & Tobias 2009). Even the $\text{Pr}_M$ dependence of the dissipation ratio has already been studied (Plunian & Stepanov 2010; Plunian et al. 2013). Their results show the inverse dissipation ratio in semilogarithmic form, so the scaling for large $\text{Pr}_M$ cannot be accurately assessed, but their results are consistent with a constant dissipation ratio for small $\text{Pr}_M$ and show a sub-linear increase at large $\text{Pr}_M$.

To assess the scaling more quantitatively, we now repeat their calculations using an independent method. The time integration is performed with an Adams-Bashforth scheme with integrating factor to treat the diffusion term exactly (Brandenburg et al. 1997). We use $N = 30$ shells for $\text{Re} = u_0/\nu k_0$ of up to $10^8$ and $\text{Pr}_M$ in the range from $10^{-6}$ to $10^8$. Forcing is applied by setting $F_1$ in Equation (16) to a complex random number at each time step, so this forcing is $\delta$-correlated, just like in the DNS. Compensated time-averaged spectra are shown in Figure 5 for three values of $\text{Pr}_M$. For $\text{Pr}_M = 1$, magnetic and kinetic energy spectra are similar, while for large (small) values of $\text{Pr}_M$, the kinetic (magnetic) energy spectrum is truncated prematurely, as is also seen in the DNS of Brandenburg (2009) for small $\text{Pr}_M$.

As mentioned above, the $\text{Pr}_M$ dependence of the dissipation ratio has already been calculated by Plunian & Stepanov (2010), and our present results agree at least qualitatively with theirs. In Figure 6 we show the $\text{Pr}_M$ dependence of the dissipation ratio $\epsilon_K/\epsilon_M$, where

$$\epsilon_K = 2\nu \sum_{n=1}^{N} k_n^2 |u_n|^2, \quad \epsilon_M = 2\eta \sum_{n=1}^{N} k_n^2 |b_n|^2. \quad (20)$$

The present shell models predict the dissipation ratio to be independent of $\text{Pr}_M$ for $\text{Pr}_M < 1$, which is in conflict with the DNS of Brandenburg (2009), where this trend was found to continue down to $\text{Pr}_M = 10^{-3}$. On the other hand, the results of Plunian & Stepanov (2010), which are overplotted in Figure 6, suggest a constant dissipation ratio only for $\text{Pr}_M \lesssim 0.01$, which is already outside the plot range of the present DNS shown in Figure 3, but still in conflict with the
For small values of $Pr_M$, the present shell models show that the kinetic energy cascade proceeds essentially independently of the magnetic field, just like in ordinary hydrodynamic turbulence. As explained in the introduction, this is also what one might have naively expected, and it is perhaps even more surprising that this is not borne out by the DNS. On the other hand, for large values of $Pr_M$, there is actually a fairly strong $Pr_M$ dependence, which is a direct consequence of $\epsilon_M$ decreasing for large $Re_M$ rather than a consequence of $\epsilon_K$ increasing. Similar results are also found for a one-dimensional passive scalar model, which will be discussed next and contrasted with a one-dimensional MHD model, which is an active scalar.

4. DISSIPATION RATIO IN DRIVEN ONE-DIMENSIONAL MODELS

The purpose of this section is to explore the possible behaviors of simple models in which the spatial extent is fully resolved, at least in one dimension. Hydrodynamics in one dimension usually involves shocks, such as the Burgers shock. The pressureless idealization of the hydrodynamic equations is known as the Burgers equation, and solutions can be found in closed form using the Cole-Hopf transformation. Before turning to the magnetic case, let us note that the evolution of a passive scalar field in the presence of a Burgers shock was already studied by Ohkitani & Dowker (2010), who found similar scaling to ours in the limit of large Schmidt number as $Sc = \nu/\kappa$, where $\kappa$ is the passive scalar diffusivity.

4.1. Passive scalar model for a Burgers shock

The passive scalar equation is a simple advection-diffusion equation given by

$$\frac{dc}{dt} = \nu \nabla^2 c,$$

where $c$ is the passive scalar concentration and a relevant passive scalar dissipation is defined as $\epsilon_c = \langle \nu | \nabla c |^2 \rangle$. For $Sc \gg 1$, Ohkitani & Dowker (2010) found $\epsilon_K/\epsilon_C \propto Sc^{1/2}$, which is in remarkable agreement with the earlier findings for hydromagnetic turbulence (Brandenburg 2009).

Specifically, the equations considered by Ohkitani & Dowker (2010) are

$$\partial u/\partial t = -uu' + \nu u''$$

$$\partial c/\partial t = -uc' + \kappa c''$$

where primes denotes differentiation with respect to $x$. The solution to Equation (22) decouples and possesses a shock. In a frame of reference moving with the shock, the solution is stationary and given by

$$u(x) = -u_0 \tanh x/w,$$

where $u_0$ is the velocity jump and $w = 2\nu/\nu_0$ is the width of the shock with $\nu = 4\nu/3$ being a rescaled viscosity. These equations can be obtained from the hydrodynamic version (i.e., $B = 0$) of Equation (2) after setting $\epsilon_s = 0$, so the density gradient does not enter, so we can ignore Equation (1) and put $\rho = 1$. The $4/3$ factor in the expression for $\nu$ comes from the fact that, owing to compressibility, the viscous acceleration term includes a $\frac{1}{3} \nabla \cdot u$ term in addition to the usual $\nu \nabla^2 u$ term; see Equation (5) for a corresponding reformulation of the dissipation terms. The viscous dissipation $\epsilon_K = \nu \int (u')^2 dx/L$ is then, using $\partial u/\partial x \propto 1/\cosh^2(x/w)$,

$$\epsilon_K = \nu \frac{w}{L} \int \frac{dx/w}{\cosh^4(x/w)} = \frac{4}{3} \frac{\nu u_0^3}{w L} = \frac{2}{3} \frac{u_0^3}{L},$$

but here the $4/3$ factor comes from the fact that $\int dx/\cosh^4 \xi = 4/3$. It is important to note that $\epsilon_K$ is constant and independent of $\nu$.

On physical grounds, the passive scalar concentration is positive definite. Mathematically, however, Equation (23) is invariant under the addition of a constant. We can therefore formulate the same boundary conditions for $c$ as for $u$, i.e., $c = \pm c_0$ and $u = \pm u_0$ for $x \to \mp \infty$, which is here truncated at finite boundary positions $x = x_\pm$ that are chosen to be sufficiently far away from the shock, i.e., $|x_\pm| \gg w$.

However, for $Sc \ll 1$, one finds $\epsilon_K/\epsilon_C \approx \text{const}$. The dependence of $\epsilon_K/\epsilon_C$ as a function of $Sc$ is shown in Figure 7, where we present the results from a numerical integration. There are clearly two different scalings for $Sc \ll 1$ and $Sc \gg 1$. The profiles of $c(x)$ are shown in Figure 8 for different values of $Sc$ and compared with the profile of $u(x)$. Not surprisingly, for small values of $Sc$, the width of the kink of $c(x)$ becomes wider.

4.2. MHD model for Alfvén kinks

An extension of the Burgers equation to MHD was already studied by Thomas (1968) and Pouquet (1993), but unlike their cases, which try to model the effects of three-dimensional dynamos, we employ here just a one-dimensional reduction of the three-dimensional equations to one dimension, which results in equations equivalent to those of Basu...
et al. (2014). This implies essentially a different sign in front of what corresponds to the stretching term in MHD, i.e., the $B \cdot \nabla B$ and $B \cdot \nabla u$ nonlinearities in the momentum and induction equations, respectively. Unlike the case of a passive scalar, the magnetic field is an active (vector) field that therefore backreacts on the flow via the Lorentz force, which in this case is just the magnetic pressure. As before, the gas pressure is neglected ($c_s = 0$), so the governing equations reduce therefore to

$$\partial u/\partial t = -uu' - bb' + \bar{v}u'', \quad (26)$$
$$\partial b/\partial t = -ub' - bu' + \eta b'', \quad (27)$$

These equations obey similar conservation equations as the full MHD equation, except that here the energy input comes from nonvanishing inflow at $x \rightarrow -\infty$ and is equal to $u_0^2/3L$. Note, however, that there is no net Poynting flux, because $ub^2 = 0$ on both boundaries.

The magnetic cases are quite different from the passive scalar case in that the magnetic field exerts a magnetic pressure. One can therefore produce a stationary state where the ram pressure of the flow from the left ($x \rightarrow -\infty$) can be balanced by the magnetic pressure of a magnetic kink when $b \rightarrow u_0$ for $x \rightarrow +\infty$ and $b \rightarrow 0$ for $x \rightarrow -\infty$. Indeed, the stationary state must obey the following system of two ordinary differential equations

$$\partial u/\partial x = (u^2 + b^2 - u_0^2)/2\bar{v}, \quad (28)$$
$$\partial b/\partial x = ub/\eta. \quad (29)$$

In practice, however, it was more straightforward to obtain solutions using direct time integration in $x_- \leq x \leq x_+$ rather than solving a two-point boundary value problem. The resulting scaling in Figure 9 confirms Equation (15) with $q \approx 0.55$ for $Pr_M > 1$ and $q \approx 0.95$ for $Pr_M < 1$.

Let us now discuss the profiles of $b(x)$ and $u(x)$ in the magnetic case. Here we find scalings that are broadly similar to those for turbulent large-scale dynamos as well as small-scale dynamos for $Pr_M < 1$, namely a slope between 0.6 and 0.7. For $Pr_M = 1$, the profiles of $b(x)$ and $u(x)$ are similar and resemble the $\tanh x/w$ profile of $u$ in the passive scalar case. However, both for $Pr_M \ll 1$ and $\gg 1$, the profiles of $b(x)$ and $u(x)$ become asymmetric, which is also the reason why we chose to integrate in a domain where $-x_- > x_+$. For small values of $Pr_M$, i.e., when $\eta \gg \nu$, the magnetic field begins to ramp up slowly and quite far away from $x = 0$. This leads to a corresponding decline of $u(x)$. On the other hand, for large values of $Pr_M$, the value of $\nu$ ($\gg \eta$) is so large that a certain imbalance of $u^2 + b^2 - u_0^2$ in Equation (28) implies only a small slope in $u(x)$, so $|u'|$ must be small.

The crucial point for the magnetic case is that the widths of the magnetic and velocity kinks are never very different from each other. Therefore, as a zeroth approximation, we can say that $\langle 2S^2 \rangle/\langle u^2 \rangle$ is approximately as large as $\langle J_0^2 \rangle/\langle B^2 \rangle$ or, in our one-dimensional case, $\langle (u')^2 \rangle/\langle u^2 \rangle$ is approximately as large as $\langle (b')^2 \rangle/\langle b^2 \rangle$. Given that in all cases $E_K \approx E_M$, this would imply that $\epsilon_K/\epsilon_M \propto \nu/\eta \approx Pr_M$, i.e., we would expect linear scaling with $Pr_M$. In this case, as we have emphasized before, the usual phenomenology of hydrodynamic turbulence, in which a decrease of $\nu$ implies a corresponding increase of dissipation, is not obeyed.

5. EFFECT OF ROTATION

The conversion of kinetic into magnetic energy is of obvious astrophysical significance. In stars with outer convection zones, a certain fraction of the kinetic energy flux is converted into magnetic energy and is observable as X-ray flux, for example (Vilhu 1984). This leads to a scaling law that has been
verified over many orders of magnitude (Christensen et al. 2009). As we have seen here, this scaling law must be affected by $Pr_M$, although the value of $Pr_M$ is approximately the same for all late-type stars, so this cannot easily be checked observationally. However, what has not been checked is whether the conversion also depends on the rotation rate.

In the work discussed in Section 2, there was no explicit rotation. Note, however, that Plunian & Stepanov (2010) did already study the effect of rotation in their shell model calculations. To check whether rotation has an influence on our results, we have performed a series of simulations with $Co = 0$, using $Pr_M = 1$ and have varied $Re_M (=Re)$ between 4 and 400, and $Co$ between 0.2 and 2. The parameters of our runs are listed in Table 2 and the result is shown in Figure 11, where we plot $\epsilon_K/\epsilon_M$ as a function of $Co$. The values of $Re_M$ are indicated by different symbols.

We see that for a given value of $Re_M$, there is a certain value of $Co = Co^*$ below which $\epsilon_K/\epsilon_M$ is roughly unaffected by rotation. As the value of $Re_M$ is increased, $Co^*$ also increases, thereby extending the range over which $\epsilon_K/\epsilon_M$ remains roughly independent of $Co$. In astrophysical applications, $Re$ is usually large enough so that we need not expect to see any rotational dependence of $\epsilon_K/\epsilon_M$. This explains why the scaling result of Christensen et al. (2009) follows the expected scaling $\epsilon_K + \epsilon_M \approx u_{rms}^3/L$ with some length scale $L$ over a huge range.

6. CONCLUSIONS

In the present work, he have extended earlier findings of a $Pr_M$ dependence of the kinetic to magnetic energy dissipation ratio, $\epsilon_K/\epsilon_M$, to the regime of small-scale and large-scale dynamos for $Pr_M > 1$ and at higher resolution than what was previously possible (Brandenburg 2011b). In most cases, our results confirm earlier results that for large-scale dynamos, the ratio $\epsilon_K/\epsilon_M$ is proportionate to $Pr_M^2$. Furthermore, we have shown that a similar scaling with $Pr_M$ can be obtained for a simple one-dimensional Alfvén kink, where ram pressure balances locally magnetic pressure. Interestingly, in these cases kinetic energy dissipation is accomplished mainly by the irrotational part of the flow rather than the solenoidal part as in the turbulence simulations presented here. We note in this connection that the kinetic energy dissipation, which is proportional to $\langle 2S^2 \rangle = \langle (\nabla \times \mathbf{u})^2 \rangle + \langle \mathbf{u} \cdot \mathbf{u} \rangle$, has similar contributions from vortical and irrotational parts.

We have also shown that for fixed values of $Pr_M$, the ratio $\epsilon_K/\epsilon_M$ is not strongly dependent on the presence of rotation, provided the magnetic Reynolds number is not too close to the marginal value for the onset of dynamo action. In the simulations with $Co = 0$ presented here, the runs were often not very long and therefore the error bars large, but the number of similar results support our conclusions that $\epsilon_K/\epsilon_M$ is roughly independent of $Co$.

For many astrophysical systems, the microscopic energy dissipation mechanism is not of Spitzer type, as assumed here. It is not obvious how this would affect our results. Nevertheless, it is clear that conclusions based on the kinetic to magnetic energy ratio itself have not much bearing on the energy dissipation ratio. This became clear some time ago in connection with local accretion disk simulations driven by the magneto-rotational instability, where magnetic energy strongly dominates over kinetic. Yet, as it turned out, most energy is dissipated viscously rather than resistively (Brandenburg et al. 1995).

Unfortunately, the question of energy dissipation is not routinely examined in astrophysical fluid dynamics, nor is it always easy to determine energy dissipation rates, because many astrophysical fluid codes ignore explicit dissipation and rely entirely on numerical prescriptions needed to dissipate energy when and where needed. Our present work highlights once again that this can be a questionable procedure, because it means that even non-dissipative aspects such as the strength
Fig. 10.— Profiles of $b(x)$ (solid) and $u(x)$ (dashed) for different values of $Pr_M$. Note that the $x$ range is the same for all panels and that we have normalized $x$ by $2u_0/(\nu + \eta)$.

The dynamo, which is characterized by $\langle u \cdot (J \times B) \rangle$, is then ill-determined. The reason why this has not been noticed earlier is that most previous work assumed $Pr_M$ to be of the order of unity. An exception is the work of Brandenburg (2009), where dynamo simulations for values of $Pr_M$ as small as $10^{-3}$ have been considered. One reason why such extreme values of $Pr_M$ have been possible is the fact that at very small values of $Pr_M$, most of the energy is dissipated resistivity, and there is not much kinetic energy left at the end of the turbulent kinetic energy cascade. As a consequence, it is then possible to decrease value of $\nu$ further and still dissipate the remaining kinetic energy, which implies that the nominal value of the fluid Reynolds number can become much larger than what is usually possible, when there is no additional resistive dissipation. However, one may wonder how a large-scale dynamo can depend on $Pr_M$. We expect that this is only possible if most of the energy transfer comes ultimately from small scales.

It is also noteworthy that there is now some evidence for non-universal behavior of the scaling of kinetic to magnetic energy dissipation ratio with $Pr_M$. Although some of the earlier results with slightly different exponents could be explained by inaccuracies and other physical effects, there are now examples such as one-dimensional simulations and the passive scalar analogy that display different exponents which cannot easily be explained through artifacts. Also, the result that for $Pr_M > 1$ the dissipation ratio scales differently in the presence of helicity ($q \approx 0.7$) than without ($q \approx 1/3$) is surprising. It would therefore be interesting to revisit the viscous to magnetic dissipation ratios over a broader range of circumstances.

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