Calculation of E and B Polarization

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1. Our Calculations

We take an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, and construct the combined field,

$$\mathbf{F} = \alpha\mathbf{E} + \beta\mathbf{B} \tag{1}$$

Now, we create the quantity,

$$P(x, y) = (F_x + iF_y)^2 \tag{2}$$

In Fourier space, we have,

$$\tilde{\mathcal{E}}(k) + i\tilde{\mathcal{B}}(k) = \tilde{P}(k)e^{-2i\psi_k}, \qquad \psi_k = \tan^{-1}\frac{k_y}{k_x} \tag{3}$$

Going back to *physical* space, we get the E and B polarizations.

1.1. A Single Function f

We take a single function $f(x, y)$ and get the fields as,

$$E_i = \partial_i f, \qquad B_i = \epsilon_{ij}\partial_j f \tag{4}$$

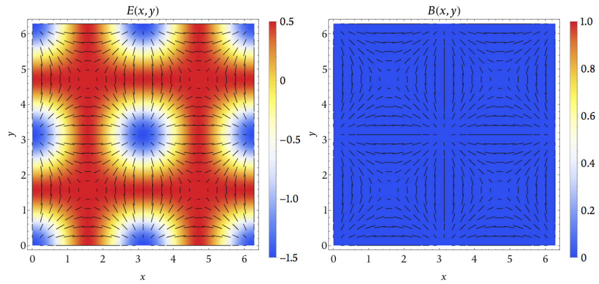
This gives us the following *real* and *imaginary* parts of $P(x, y)$,

$$\begin{aligned} \mathrm{Re}\left[P(x,y)\right] &= \left(\alpha^2 - \beta^2\right)\left[(\partial_x f)^2 - (\partial_y f)^2\right] + 4\alpha\beta\left[(\partial_x f)(\partial_y f)\right] \\ \mathrm{Im}\left[P(x,y)\right] &= 2\left[\left(\alpha^2 - \beta^2\right)(\partial_x f)(\partial_y f) + \alpha\beta\left\{(\partial_y f)^2 - (\partial_x f)^2\right\}\right] \end{aligned} \tag{5}$$

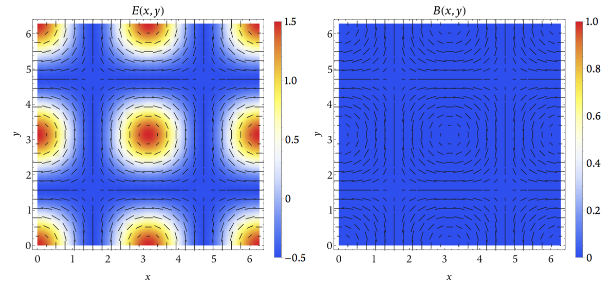
We can see that the cases $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$ differ only by a sign, as far as $P(x, y)$ is concerned. And so, with the choice,

$$f(x, y) = \cos x \cos y \tag{6}$$

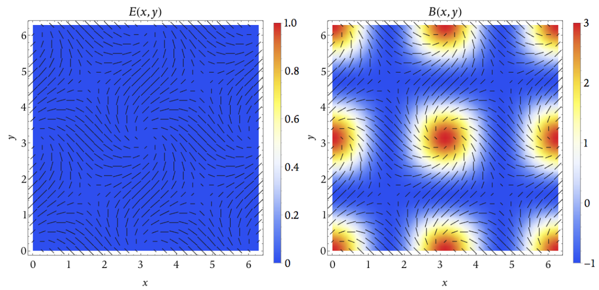
we get the polarization patterns in Fig.1.



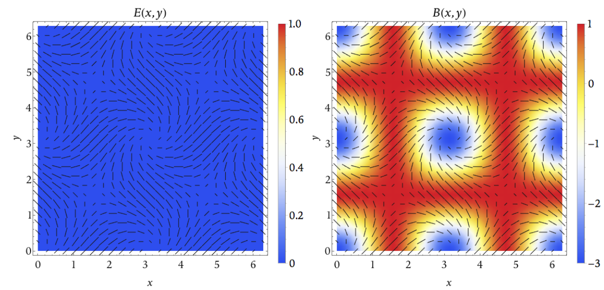
(a) $\alpha = 1, \beta = 0$



(b) $\alpha = 0, \beta = 1$



(c) $\alpha = 1, \beta = 1$



(d) $\alpha = 1, \beta = -1$

Figure 1: E and B Polarization when $\mathbf{E}$ and $\mathbf{B}$ are both derived from a single function $f(x, y)$.

1.2. Two Functions f and g

If have two functions $f(x, y)$ and $g(x, y)$, such that,

$$E_i = \partial_i f, \qquad B_i = \epsilon_{ij}\partial_j g \tag{7}$$

we get,

$$\begin{aligned}\mathrm{Re}\,[P(x,y)] &= \alpha^2\Big[(\partial_x f)^2 - (\partial_y f)^2\Big] - \beta^2\Big[(\partial_x g)^2 - (\partial_y g)^2\Big] + 2\alpha\beta\Big[(\partial_x f)(\partial_y g) + (\partial_x g)(\partial_y f)\Big] \\ \mathrm{Im}\,[P(x,y)] &= 2\Big[\alpha^2(\partial_x f)(\partial_y f) - \beta^2(\partial_x g)(\partial_y g) + \alpha\beta\Big\{(\partial_y f)(\partial_y g) - (\partial_x f)(\partial_x g)\Big\}\Big]\end{aligned} \tag{8}$$

In this case, with,

$$f(x, y) = \cos x \cos y, \qquad g(x, y) = \sin x \sin y \tag{9}$$

we get the polarization patterns in Fig.2.

The bottom row of Fig.2 is a bit strange. However, if we skew the ratioo a bit ($\alpha = 1, \beta = 4$), we get something like Fig.3.

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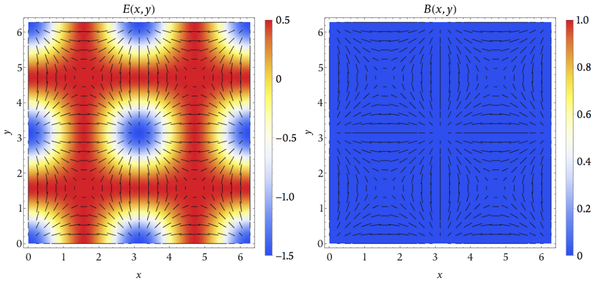
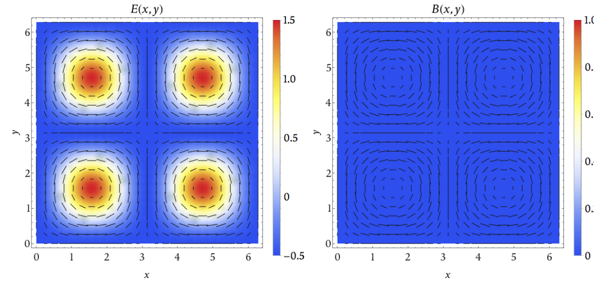
2. Formulae in Durrer Book

Let us start with eq.(1) where the fields are given either by eq.(7) (and by eq.(4)). We now construct the tensor,

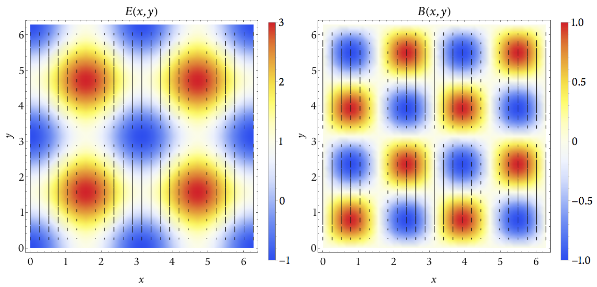
$$\begin{aligned}P_{ij} = F_i F_j &= \big(\alpha E_i + \beta B_i\big)\big(\alpha E_j + \beta B_j\big) \\ &= \alpha^2 E_i E_j + \alpha\beta(E_i B_j + B_i E_j) + \beta^2 B_i B_j\end{aligned} \tag{10}$$



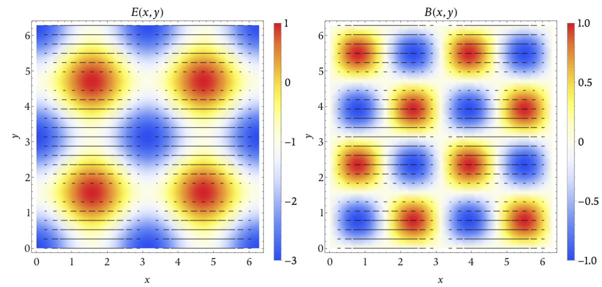
(a) $\alpha = 1, \beta = 0$



(b) $\alpha = 0, \beta = 1$



(c) $\alpha = 1, \beta = 1$



(d) $\alpha = 1, \beta = -1$

Figure 2: E and B Polarization when **E** and **B** are both derived from respectively from $f(x, y)$ and $g(x, y)$.

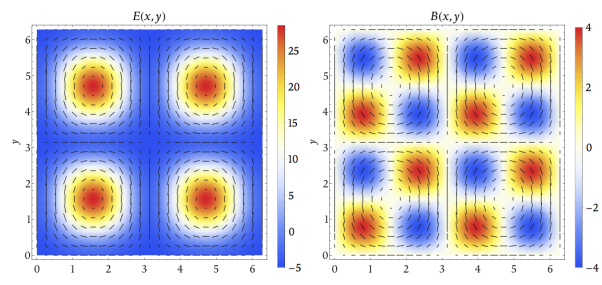


Figure 3: E and B Polarization when **E** and **B** are both derived from respectively from $f(x, y)$ and $g(x, y)$, and $\alpha = 1, \beta = 4$.

We now plan to use **eq. (5.83)** in Durrer's book to calculate $\mathcal{E}$ and $\mathcal{B}$ ($\tilde{\mathcal{E}}$ and $\tilde{\mathcal{B}}$ in the book.)

$$
\begin{aligned}
\mathcal{E} &= 2\,\mathrm{div}\,\mathrm{div}\,P = 2\partial_i\partial_j P_{ij} = 2\partial_i\partial_j\Big[\alpha^2 E_iE_j + \alpha\beta\left(E_iB_j + B_iE_j\right) + \beta^2 B_iB_j\Big] \\
&= 2\partial_i\Big[\alpha^2\Big\{(\partial_j E_i)E_j + E_i(\partial_j E_j)\Big\} + \beta^2\Big\{(\partial_j B_i)B_j + B_i(\partial_j B_j)\Big\} \\
&\quad + \alpha\beta\Big\{(\partial_j E_i)B_j + E_i(\partial_j B_j) + (\partial_j B_i)E_j + B_i(\partial_j E_j)\Big\}\Big] \\
&= 2\alpha^2\Big[(\partial_i\partial_j E_i)E_j + E_i(\partial_i\partial_j E_j) + (\partial_j E_i)(\partial_i E_j) + (\partial_i E_i)^2\Big] \\
&\quad + 2\beta^2\Big[(\partial_i\partial_j B_i)B_j + B_i(\partial_i\partial_j B_j) + (\partial_j B_i)(\partial_i B_j) + (\partial_i B_i)^2\Big] \\
&\quad + 2\alpha\beta\Big[(\partial_i\partial_j E_i)B_j + (\partial_j E_i)(\partial_i B_j) + (\partial_i E_i)(\partial_j B_j) + E_i(\partial_i\partial_j B_j) \\
&\qquad + (\partial_i\partial_j B_i)E_j + (\partial_j B_i)(\partial_i E_j) + (\partial_i B_i)(\partial_j E_j) + B_i(\partial_i\partial_j E_j)\Big]
\end{aligned}
$$

Using the facts,

$$\partial_i \partial_j \equiv \partial_j \partial_i, \qquad \nabla \cdot \mathbf{B} = 0$$

we can considerably simplify the above expression,

$$\begin{aligned} \mathcal{E} = & 2\alpha^2 \Big[(\partial_i \partial_j E_i) E_j + E_i (\partial_i \partial_j E_j) + (\partial_j E_i)(\partial_i E_j) + (\partial_i E_i)^2 \Big] + 2\beta^2 (\partial_j B_i)(\partial_i B_j) \\ & + 2\alpha\beta \Big[(\partial_i \partial_j E_i) B_j + (\partial_j E_i)(\partial_i B_j) + (\partial_j B_i)(\partial_i E_j) + B_i (\partial_i \partial_j E_j) \Big] \end{aligned} \tag{11}$$

2.1. Two Functions f and g

We consider the expressions term by term:

- The β^2 term:

$$(\partial_j B_i)(\partial_i B_j) = (\epsilon_{im} \partial_j \partial_m f)(\epsilon_{im} \partial_j \partial_m f)$$

  Using $\epsilon_{im}\epsilon_{jn} = \delta_{in}\delta mn - \delta_{in}\delta_{jm}$, we get,

$$\beta^2 (\partial_j B_i)(\partial_i B_j) = \beta^2 \Big[(\partial_i \partial_m g)(\partial_i \partial_m g) - (\nabla^2 g)^2 \Big] \tag{12}$$

- The $\alpha\beta$ term:
- The α^2 term: