# Numerical study of an axisymmetric anisotropic conductivity dynamo

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## Abstract

Plunian & Alboussière (2020) have recently presented an analytic solution to an axisymmetric dynamo for an anisotropically conducting medium. The velocity field corresponds to that of a rigidly rotating cylinder embedded in a conductor at rest. Here we present numerical simulations for smooth angular velocity profiles. We also consider the case where the anisotropy is confined to be within the rotating cylinder only.

## 1 Introduction

A major obstacle in dynamo theory has been the fact that the magnetic field must always be non-axisymmetric and therefore three-dimensional. But this is not strictly true if an allows for the conductivity to be anisotropic. For that case, Plunian & Alboussière (2020) found recently a solution which is extremely remarkable in many respects. (i) Using cylindrical coordinates, $(r, \phi, z)$, the three components of the magnetic field depend only on radius $r$ and height $z$, but not on the azimuthal angle $\phi$. (ii) The velocity field is very simple, corresponding to a rigidly rotating cylinder embedded in a conducting solid at rest. (iii) The growth rate increases with increasing conductivity without limit and does not go to zero as for slow dynamos. (iv) The solution is given in closed form.

Understanding the properties of such a dynamo could be of interest not only for the realization of laboratory dynamos, but also for dynamos in nearly collisionless plasmas, where the vastness of the full six-dimensional phase space constitutes a significant technical challenge. It should be kept in mind, however, that the experimental realization of such an anisotropically conducting medium may, in practice, involve a nonaxisymmetric design. This was already discussed by Plunian & Alboussière (2020), who suggested a composite of conducting layers arranged in a logarithmic spiral interlaced with poorly contacting or insulating layers.

The connection between dynamos in a medium of anisotropic conductivity and the homopolar disk dynamo was already pointed out by Ruderman & Ruzmaikin (1984). They considered a two-dimensional dynamo in Cartesian coordinates, for which Alboussière et al. (2020) found an analytic solution. A feasible design of the homopolar disk dynamo was presented by Priede & Avalos-Zúñiga (2020).

Plunian & Alboussière (2020) emphasize that the rotation must be retrograde. Dynamo action requires then $\alpha > 0$. For $\alpha < 0$, dynamo action occurs for prograde rotation.

## 2 The model

We adopt rigid rotation with angular velocity $\Omega_0$ in the inner cylinder of radius $R$, with the material at rest outside, i.e.,

$$\Omega(r) = \Omega_0 \, f(r) \tag{1}$$

where $\Omega_0$ is a constant and $f(r)$ is a radial profile modelling the change from finite to vanishing angular velocity in a continuous fashion, given by

$$f(r) = 1 - \Theta(r, d). \tag{2}$$

We use the same profile function to also allow for a radial truncation of the anisotropy using

$$\eta_{ij} = \eta_0 \, \delta_{ij} + \eta_1(r) \, q_i q_j \tag{3}$$

where $\eta_1 = \eta_{10} f(r)$. We solve the uncurled induction equation,

$$\frac{\partial A_i}{\partial t} = (\boldsymbol{u} \times \boldsymbol{B})_i - \eta_{ij} J_j \tag{4}$$

We define a magnetic Reynolds number as

$$R_\Omega = \Omega_0 R^2/\eta_0, \tag{5}$$

and the degree of anisotropy as

$$s = \eta_{10}/\eta_0. \tag{6}$$

Plunian & Alboussière (2020) found dynamo action for $|R_\Omega| > 14.61$ for $s \to \infty$ using the optimal angle $\alpha = 0.16\pi \approx 29°$ and wavenumber $kR = 1.1$, corresponding to a minimal height of $L_z/R = 5.7$.

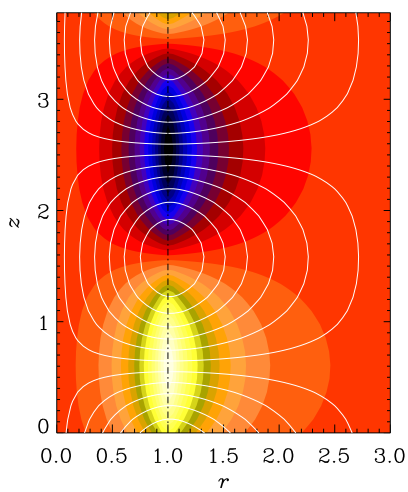


Figure 1: `psav_64x32r1` Anisotropic conductor both in the interior and the exterior. Here, $\eta_0 = 0.0440$, $L_r = 3$, and $L_z = 3.9$

# 3 Results

In Figure 1 we show the result for the case where $\eta_1 = \text{const}$. The critical maximal magnetic diffusivity is $\eta_0 = 0.0440$, corresponding to $R_\Omega = 22.7$, and the vertical wavenumber is $kR = 1.6$. In Figure 2 we show the case where $\eta_1 \to 0$ for $r > R$. The critical maximal magnetic diffusivity is $\eta_0 = 0.0218$, corresponding to $R_\Omega = 45.9$, and the vertical wavenumber is $kR = 1.7$.

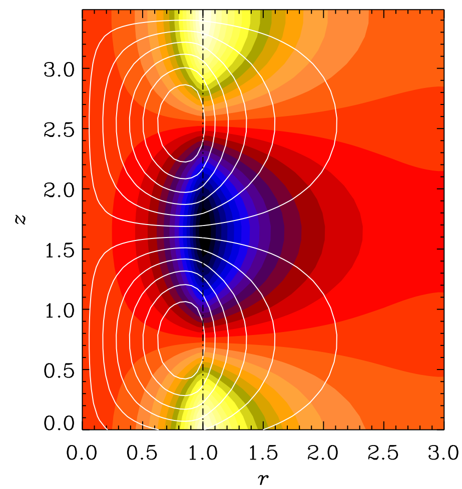


Figure 2: `psav_I64x32k4` Isotropic conductor in the exterior, outside the cylinder. Here, $\eta_0 = 0.0218$, $L_r = 3$, and $L_z = 3.6$

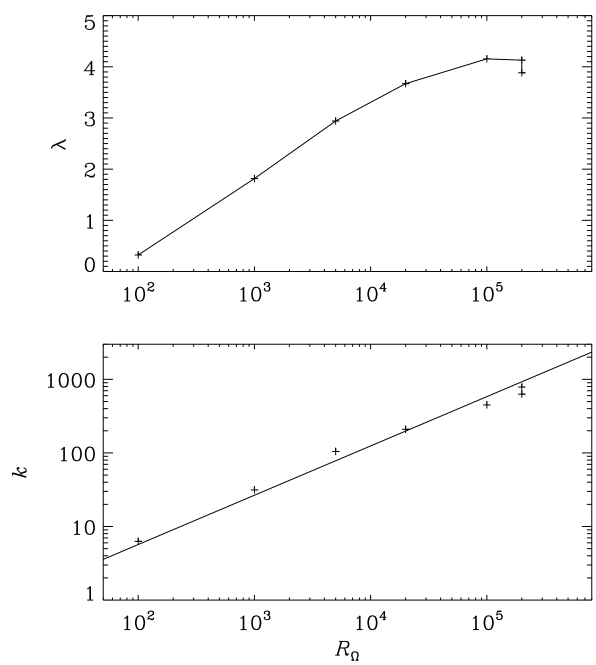


Figure 3: `p` Growth rate and k.

# References

Alboussière, T., Drif, K., & Plunian, F., Dynamo action in sliding plates of anisotropic electrcal conductivity. *Phys. Rev. E* 2020, **101**, 033107.

Plunian, F., & Alboussière, T., Axisymmetric dynamo action is possible with anisotropic conductivity. *Phys. Rev. Research* 2020, **2**, 013321.

Priede, J., & Avalos-Zúñiga, R., Feasible homopolar dynamo with sliding liquid-metal contacts. *Phys.*

Table 1: Models with $\eta_1/\eta_0 = 5$, $\alpha = 28.8°$, and $kR = 1.1$.

| $\eta_0$ | $\eta_1$ | $\lambda$ | $k$ |
|---|---|---|---|
| 0.0445 | 0.0035 | | |
| 0.04607 | -0.0004 | | |
| 0.0467 | -0.0014 | | |

Table 2: Models with $\eta_1/\eta_0 = 5$, $\alpha = 30°$, and $kR = 1.1$.

| $\eta_0$ | $L_z$ | $\lambda$ |
|---|---|---|
| $1 \times 10^{-2}$ | 1 | 0.3221 |
| $1 \times 10^{-3}$ | 0.2 | 1.816 |
| $2 \times 10^{-4}$ | 0.06 | 2.9404 |
| $5 \times 10^{-5}$ | 0.03 | 3.6706 |
| $1 \times 10^{-5}$ | 0.03 | 3.6706 |
| 1e-5 | 0.010 | 3.3899 |
| 1e-5 | 0.012 | 3.9734 |
| 1e-5 | 0.014 | 4.155 |
| 1e-5 | 0.016 | 4.1450 |
| 1e-5 | 0.018 | 4.0539 |
| 5e-6 | 0.010 | 4.130 |
| 5e-6 | 0.008 | 3.883 |

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Ruderman, M. S., & Ruzmaikin, A. A., Magnetic field generation in an anisotropically conducting fluid. *Geophys. Astrophys. Fluid Dynam.* 1984, **28**, 77–88.

Table 3: Models with $\eta_1 = 0$ for $r \gg 1$ and $\eta_{10}/\eta_0 = 5$.

| $\eta_0$ | $L_r$ | $L_z$ | $\lambda$ |
|---|---|---|---|
| 0.010 | 2 | 2.0 | 0.0886 |
| 0.014 | 2 | 2.0 | 0.0441 |
| 0.016 | 2 | 2.0 | 0.0208 |
| 0.018 | 2 | 2.0 | $-0.0030$ |
| 0.018 | 2 | 2.1 | $+0.0024$ |
| 0.018 | 2 | 2.2 | $+0.0066$ |
| 0.018 | 2 | 2.4 | $+0.0123$ |
| 0.018 | 2 | 2.6 | $+0.0155$ |
| 0.018 | 2 | 2.8 | $+0.0171$ |
| 0.018 | 2 | 3.0 | $+0.0177$ |
| 0.018 | 2 | 3.2 | $+0.0176$ |
| 0.020 | 2 | 3.2 | $+0.0071$ |
| 0.022 | 2 | 3.2 | $-0.0035$ |
| 0.022 | 2 | 3.0 | $-0.0059$ |
| 0.022 | 2 | 3.3 | $-0.0027$ |
| 0.022 | 2 | 3.4 | $-0.0021$ |
| 0.022 | 2 | 3.5 | $-0.0016$ |
| 0.022 | 2 | 3.6 | $-0.0012$ |
| 0.022 | 2 | 3.7 | $-0.0009$ |
| 0.022 | 2 | 3.8 | $-0.0007$ |
| 0.022 | 2 | 3.9 | $-0.0006$ |
| 0.022 | 2 | 4.0 | $-0.0006$ |
| 0.022 | 3 | 4.0 | $-0.0002$ |
| 0.022 | 4 | 4.0 | $-0.0011$ |
| 0.022 | 3 | 4.2 | $-0.0007$ |
| 0.022 | 3 | 3.8 | $-0.0002$ |
| 0.0216 | 3 | 3.8 | $+0.0014$ |
| 0.0216 | 3 | 3.7 | $+0.0014$ |
| 0.0216 | 3 | 3.6 | $+0.0013$ |
| 0.0218 | 3 | 3.6 | $+0.0001$ |

Table 4: Models with $\eta_1 = 0$ for $r \gg 1$ and $\eta_{10}/\eta_0 = 5$.

| $d_\Omega$ | $d_\eta$ | $\eta_0$ | $L_r$ | $L_z$ | $\lambda$ |
|---|---|---|---|---|---|
| 0.02 | 0.02 | 0.0218 | 3 | 3.6 | $+0.0001$ |
| 0.05 | 0.02 | 0.0218 | 3 | 3.6 | $-0.0013$ |
| 0.05 | 0.05 | 0.0218 | 3 | 3.6 | $+0.0034$ |

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Table 5: Models with anisotropy throughout, i.e., $\eta_1 = \eta_{10} = 5\,\eta$.

| $\eta_0$ | $L_r$ | $L_z$ | $\lambda$ |
|---|---|---|---|
| 0.0218 | 3 | 1.8 | +0.1010 |
| 0.0300 | 3 | 1.8 | −0.0082 |
| 0.0300 | 3 | 2.0 | +0.0051 |
| 0.0300 | 3 | 2.2 | +0.0292 |
| 0.0294 | 3 | 1.8 | −0.0080 |
| 0.0294 | 3 | 2.2 | +0.0360 |
| 0.0306 | 3 | 2.2 | +0.0220 |
| 0.0312 | 3 | 2.2 | +0.0146 |
| 0.0324 | 3 | 2.2 | +0.0018 |
| 0.0324 | 3 | 2.4 | +0.0197 |
| 0.0340 | 3 | 2.6 | +0.0197 |
| 0.0360 | 3 | 2.8 | +0.0148 |
| 0.0380 | 3 | 3.0 | +0.0095 |
| 0.0400 | 3 | 3.2 | +0.0042 |
| 0.0410 | 3 | 3.2 | −0.0018 |
| 0.0410 | 3 | 3.3 | +0.0017 |
| 0.0410 | 3 | 3.4 | +0.0046 |
| 0.0420 | 3 | 3.5 | +0.0018 |
| 0.0425 | 3 | 3.6 | +0.0016 |
| 0.0430 | 3 | 3.7 | +0.0013 |
| 0.0435 | 3 | 3.8 | +0.0008 |
| 0.0440 | 3 | 3.9 | +0.0002 |