1. Stability of different solution branches

Consider the following amplitude equation:

\[ \dot{\xi} = (R - R_c)\xi + \epsilon \xi^2 - \xi^3 \]  

(1)

Here, \( \xi \) is the amplitude, \( R \) is the control parameter, and \( R_c \) is the critical value for onset. Consider the parameters \( R_c = 1 \) and \( \epsilon = 1 \).

(a) Determine all fixed points of Eq. (1). Plot \( \xi \) vs. \( R \). At this point, use only a dotted line for each of the solution branches. What is the nature of the bifurcation at \( R = R_c \)? (Is it super- or subcritical?)

(b) For each of these fixed points, linearize Eq. (1) around these fixed points and determine thereby the stability of these solutions to the full nonlinear equation on all branches. Now modify your plot and mark all stable solutions as a fat line.

(c) Compute numerically the time dependence \( \xi(t) \) for \( R = 0.8 \) using as initial conditions \( \xi(0) = 0.2764, 0.2763, 0.72, \) and \( 0.74 \). For each of the four cases, plot (i) \( \xi = \xi(t) \), (ii) \( |\xi(t) - \xi_0| \), where \( \xi_0 \) is the relevant fixed point solution, and finally (iii) the instantaneous growth rate, \( \sigma(t) = \frac{d}{dt} \ln |\xi(t) - \xi_0| \), and determine the time interval in which \( \sigma(t) \) can be used to estimate the growth or decay rate of the perturbed solution.

(d) Compare the \( \sigma(t) \) with the \( \sigma \) obtained under (b). Explain in a few words the reasons for discrepancies. Also, explain in a few words the structure of the bifurcation diagrams in terms of the slope of the branches. What do you think one should do with negative values of \( \xi \)? What happens if \( \epsilon \) were negative?

(e) Imagine applying the amplitude equation to a nonlinear laboratory dynamo problem, where dynamo action occurs when the magnetic Reynolds number has to be above a certain critical value for the dynamo to be excited. In the lab you would measure \( \approx 0.5 \text{ G} \) even in the absence of a dynamo because of the Earth’s magnetic field. How do you think Eq. (1) needs to be modified to take this into account.

2. Finite difference formula. Finite derivative formulae for the \( n \)th derivative can generally be written as

\[ \frac{d^n f_i}{dx^n} = \frac{1}{\delta x^n} \sum_{j=-N}^{N} c_j^{(n)} f_{i+j}, \]

(2)

with coefficient \( c_j^{(n)} \) given in Table 1 for schemes of order \( N \).

To derive the coefficients \( c_j^{(n)} \), consider the Taylor expansion for \( f_{i+j} \equiv f_i(j \delta x) \),

\[ f_{i+j} = f_i + j \delta x f_i' + \frac{1}{2} (j \delta x)^2 f_i'' + \frac{1}{6} (j \delta x)^3 f_i''' + \frac{1}{24} (j \delta x)^4 f_i^{(4)} + \frac{1}{120} (j \delta x)^5 f_i^{(5)} + ... \]

(3)

(a) Write the Taylor expansion for \( f_{i+j} \) as matrix equation

\[ f_{i+j} = M_{jk} \left( \frac{d}{dx} \right)^k f_i, \]

(4)
Table 1: Coefficients $c_j^{(n)} = a_j^{(n)}/b^{(n)}$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$n$</th>
<th>$b^{(n)}$</th>
<th>$a_0^{(n)}$</th>
<th>$a_1^{(n)}$</th>
<th>$a_2^{(n)}$</th>
<th>$a_3^{(n)}$</th>
<th>$a_4^{(n)}$</th>
<th>$a_5^{(n)}$</th>
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<tbody>
<tr>
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<td>1</td>
<td>2520</td>
<td>0</td>
<td>2100</td>
<td>−600</td>
<td>150</td>
<td>−25</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>840</td>
<td>0</td>
<td>672</td>
<td>−168</td>
<td>32</td>
<td>−3</td>
<td></td>
</tr>
<tr>
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<td>1</td>
<td>60</td>
<td>0</td>
<td>45</td>
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<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>12</td>
<td>0</td>
<td>8</td>
<td>−1</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
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</tr>
<tr>
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<td>25200</td>
<td>−73766</td>
<td>42000</td>
<td>−6000</td>
<td>1000</td>
<td>−125</td>
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</tr>
<tr>
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<td>8064</td>
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<tr>
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<tr>
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<td>1</td>
<td>−2</td>
<td>1</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

where $M_{jk} = (j \delta x)^2$ is a matrix whose rank depends on the order of the scheme. Invert the matrix to compute the coefficients in front of the $f_{i+j}$ for the finite difference derivative

$$
\left(\frac{d}{dx}\right)^k f_i = (M_{jk})^{-1} f_{i+j}
$$

and verify the values given in Table 1.

(b) Extend the table to compute the coefficients for the third derivative with a stencil width 5.

(c) What is the error of these schemes, i.e., what is the power of $\delta x$ with which it scales, what is the leading derivative, and what are the coefficients. One or two examples of your choice will be enough.

3. Second-next nearest neighbor shell model. In Lecture 10, a shell model with nearest neighbors was presented. A shell model with second-next nearest neighbors allows us to conserve two conservation laws. The models are sometimes called GOY models to acknowledge the work of Gledzer, Ohkitani, and Yamada.

(a) Show that the general form of such a model is

$$
\frac{du_n}{dt} = ik_n (A_{n-2}u_{n-1} + B_{n-1}u_{n+1} + C_{n+1}u_{n+2})^* - \nu k_n^2 u_n.
$$

where $k_n = k_0 2^n$ is the wavenumber shell. The asterisk means complex conjugation.

(b) Assume that both energy $E = \frac{1}{2} \sum |u_n|^2$ and enstrophy $\Xi = \frac{1}{2} \sum k_n^2 |u_n|^2$ are conserved, and show that

$$
A = -\frac{1}{10}, \quad B = 1, \quad C = -\frac{8}{5}.
$$

(c) Next, assume that both energy $E = \frac{1}{2} \sum u_n^2$ and helicity $H = \frac{1}{2} \sum (-1)^n k_n |u_n|^2$ are conserved, and show that

$$
A = \frac{1}{2}, \quad B = 1, \quad C = -4.
$$