1. Picking up some pieces.

(a) For a monatomic gas, the stress tensor is \( \mathbf{T} = 2\rho v \mathbf{S} \) with \( S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{u} \).

Show that

\[
\tau_{ij} \cdot u_j = \tau : \nabla \mathbf{u} = 2\rho v S^2. \tag{1}
\]

Hint: make use of the facts that (i) \( \mathbf{S} \) is a symmetric tensor, and (ii) it is trace-free.

(b) Furthermore, show that in this case, the viscous acceleration is

\[
\nabla \cdot \mathbf{\tau} = \rho \nabla (\nabla^2 \mathbf{u} + \frac{1}{3} \nabla \nabla \cdot \mathbf{u} + 2 \nabla \ln \rho v). \tag{2}
\]

(c) Show that, for an ideal gas,

\[
\rho T \frac{D_S}{Dt} = \rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt}, \tag{3}
\]

Hint: use the facts that (i) \( R/\mu = c_p - c_v \), \( D_s = c_v D \ln p - c_p D \ln \rho \), and \( D \ln p = D \ln T + D \ln \rho \).

Note: there is really a \( c_p \) factor in front of the \( DT/Dt \) term, not \( c_v \).

2. Momentum and energy equations in conservative forms. Consider the continuity, momentum, and energy equations in the form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{4}
\]

and

\[
\frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \tag{5}
\]

\[
\frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e + \frac{p}{\rho} \nabla \cdot \mathbf{u} = 0, \tag{6}
\]

where \( e \) is the internal energy per unit mass.

(a) Derive the evolution equation for the momentum density

\[
\frac{\partial}{\partial t} \left( \frac{2}{3} \rho \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial}{\partial x_j} \left( \rho \frac{\partial u_i}{\partial x_j} \right) - \frac{\partial}{\partial x_j} ( \rho u_i u_j + \delta_{ij} p ) + \frac{p' \rho}{\rho} \frac{\partial \phi}{\partial x_j}, \tag{7}
\]

Note that summation over double indices is assumed!

(b) Explain why this equation is in *conservative* form. Discuss how the volume-integrated momentum changes for periodic boundary conditions. What other boundary conditions give the same result?

(c) Derive the so-called total energy equation in the form

\[
\frac{\partial}{\partial t} ( \frac{1}{2} \rho u^2 + \rho e ) = -\frac{\partial}{\partial x_j} \left[ u_j \left( \frac{1}{2} \rho u^2 + \rho e + \frac{p'}{\rho} \right) \right], \tag{8}
\]

Again, summation over double indices is assumed.
(d) Explain in words how these equations can be used to say something about hydrodynamic planar shocks, where density, pressure, and velocity can change discontinuously across a surface. Consider a one-dimensional frame of reference comoving with the shock. What happens to the time derivative in that frame? Use the equation of state in the form

\[ p = (\gamma - 1)\rho e \]

and count how many unknowns do you have?

(e) Next, restore the dissipation term \( \nabla \cdot \tau_{ij} \) on the rhs of Eq. (5) for \( \partial u_i / \partial t \) and show that Eq. (6) must then attain a corresponding heating term, \( u_i \tau_{ij} \) on the right hand side. Include these terms in Eqs. (7) and (8).

3. Numerical sound wave experiments. Familiarize yourself with the experimental setup https://www.nordita.org/~brandenb/teach/PencilCode/NonlinearSound.html of the PENCIL CODE experiment using the initial condition:

\[
\begin{align*}
&\text{&hydro_init_pars} \\
&\quad \text{inituu='sound-wave', ampluu=1e-3} \\
&\text{&density_init_pars} \\
&\quad \text{initlnrho='sound-wave', ampllnrho=1e-3} \\
&\text{&entropy_init_pars} \\
&\quad \text{}/ \\
\end{align*}
\]

This corresponds to \( u = (u_x, 0, 0) \) with \( u_x = A_u c_s \sin kx \) and \( \ln \rho = A_\rho \sin kx \), where \( c_s = 1 \) by default, and \( A_u = A_\rho = 10^{-3} \) is chosen. Also, by default, \( s = 0 \).

(a) Write down (and verify) the time-dependent solution of the linearized continuity, momentum, and energy equations for the initial condition specified above.

(b) What happens as time goes on? Does the wave travel? If so, in which direction?

(c) Determine empirically the speed with which the initial sine wave propagates. Describe your working in all detail.

(d) Modify the initial condition such that \( A_u = A_\rho = 1 \). Describe your working in all detail. Present an \( xt \) diagram of \( u_x \) and \( \ln \rho \).

(e) Repeat the experiment with \( \nu = 0.05 \). Does the speed of the wave change? Does the amplitude change? Compare with your theoretical expectation. Explain your working. To iterate upon your answer, you may want to consider trying \( \text{ivisc='simplified'} \) in \text{viscosity_run_pars}, or \( \text{inituu='sinwave-phase', ampluu=1e-3, } \text{ampuy=1} \) in \text{hydro_init_pars}.

(f) Change the amplitude to unity, i.e., put \( A_u = A_\rho = 1 \). Describe what happens. By how much does the kinetic energy decrease. What happens to the thermal energy?

(g) Can you get shocks with with \( A_u = A_\rho = 10^{-3} \)? You will need to increase the value of \( \text{tmax} \) to 2000 to find a fair answer.

1 In the PENCIL CODE, by default, the pressure gradient is written as \( \rho^{-1} \nabla p = c_s^2 (\nabla s/c_p + \nabla \ln \rho) \). Here, \( c_s \) is the sound speed, given by \( c_s^2 = \gamma p/\rho = c_s^2_0 \exp[\gamma/2 + (\gamma - 1) \ln(\rho/\rho_0)] \) with \( c_s^2_0 = \ln \rho_0 = 1 \); see page 61 of the manual, http://pencil-code.nordita.org/doc/manual.pdf. Use 256 mesh points, and, to begin with, zero viscosity \( \nu = 0 \).
4. Sound waves in an isothermally stratified atmosphere. Consider the continuity and momentum equations for an isothermal atmosphere (constant temperature) and an isothermal equation of state (special case with $\gamma = 1$) with constant speed of sound, $c_s$, and uniform gravity, $g$, in one dimension,

\[
\frac{\partial \rho}{\partial t} + u_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial u_z}{\partial z} = 0, \quad (9)
\]

\[
\rho \frac{\partial u_z}{\partial t} + \rho u_z \frac{\partial u_z}{\partial z} + c_s^2 \frac{\partial \rho}{\partial z} + \rho g = 0, \quad (10)
\]

where $\rho$ is density and $u_z$ vertical velocity.

(a) Show that these equations obey an equilibrium solution $u_z = u_{z0}(z)$, $\rho = \rho_0(z)$, given by

\[
u_{z0}(z) = 0, \quad \rho_0(z) = \rho_{00} e^{-z/H}, \quad (11)
\]

where $\rho_{00}$ is a constant and $H = c_s^2/g$ is the vertical scale height.

(b) Write $\rho = \rho_0 + \rho_1$ and $u_z = u_{z1}$ and linearize equations (9) and (10) with respect to $\rho_1$ and $u_{z1}$.

(c) Assume that $\rho_1$ and $u_{z1}$ take the form

\[
\rho_1(z, t) = \hat{\rho}_1 e^{ikz-i\omega t-z/2H}, \quad (12)
\]

\[
u_{z1}(z, t) = \hat{u}_{z1} e^{ikz-i\omega t+z/2H}, \quad (13)
\]

and show that the linearized equations can be written as

\[
\left( \begin{array}{cc}
-i\omega & -c_s^2(k-(2H)^{-1}) \\
[ik+(2H)^{-1}]c_s^2 & -i\omega
\end{array} \right) \left( \begin{array}{c}
\hat{\rho}_1 \\
\rho_{00}\hat{u}_{z1}
\end{array} \right) = \left( \begin{array}{c}
0 \\
0
\end{array} \right) \quad (14)
\]

(d) Calculate the dispersion relation. Note: it will be convenient to use the abbreviation $\omega_0 = c_s/2H$ for the acoustic cutoff frequency.

(e) Give a qualitative plot of the dispersion relation.

(f) Calculate the value of the period $2\pi/\omega_0$ for the solar atmosphere, assuming $c_s = 6 \text{ km s}^{-1}$ and $g = 270 \text{ m/s}^2$, and the Earth’s atmosphere, assuming $c_s = 300 \text{ m/s}$ and $g = 10 \text{ m/s}^2$. 
