Lecture 9: Basic aspects of turbulence and turbulent structures

The definition of what is turbulence is difficult. Turbulent flows are irregular in space and time and they typically extend over broad length and time scales. The standard case is vortical turbulence, but there is in principle also irrotational (or so-called acoustic) turbulence.

Turbulence is often caused by instabilities. In many theoretical studies of turbulence, turbulence is driven by volume forcing instead. This has the advantage of simulating homogeneous turbulence. (Convection driven by the Rayleigh-Benard or the magneto-rotational instability is not homogeneous.) However, turbulence can also be decaying and is caused just by a sufficiently irregular initial condition. An excellent textbook on turbulence is that by Davidson (2015).

1 Appearance of turbulence

The standard picture of bigger vortices breaking up into smaller ones is misleading. With the advance of computer simulations, it became clear that turbulence consists instead of tubes; see Figure 1.

![Figure 1: Examples of vortex tubes in homogeneous turbulence from She et al. (1990) (left panel) and Porter et al. (1998) (right panel).](image)

The left-hand panel of Figure 1 shows examples of vortex tubes. Their thickness is related to the viscous scale while their length was often expected to be comparable with the integral scale. However, in subsequent years simulations at increasingly higher Reynolds numbers seem to reveal that the vortex turbulence become a less prominent feature of otherwise nebulous looking structures of variable density (see the right-hand panel of Figure 1).

The production of vortex tubes is associated with the action of local strain and thus with the strain tensor \( s_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \). It is best characterized by its principle axes given by the three eigenvectors, \( e_1 \), \( e_2 \), and \( e_3 \), corresponding to the eigenvalues \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_3 \). Their sum is equal to \( \nabla \cdot \mathbf{u} \), and vanishes for incompressible flows. The largest one is corresponds to the direction of stretching, and the smallest one to the direction of compression; see Figure 2.

It has been known for some time that in isotropic turbulence the vorticity vector tends to be aligned with the direction \( \hat{e}_2 \) and is therefore normal to the plane where the flow would be two-dimensional. If the turbulence was perfectly two-dimensional, the intermediate eigenvalue of the rate-of-strain tensor would vanish. This is however not the case; see Figure 3 where we plot probability density functions (PDFs) of the three eigenvalues.
Figure 2: Sketch illustrating the principle axes in a local shear flow.

Figure 3: PDF of the eigenvalues of the rate-of-strain tensor. Note that the intermediate ones are not vanishing, as expected for two-dimensional turbulence.

2 Spectra

Incompressible forced turbulence simulations have been carried out at resolutions in excess of 4096^3 meshpoints [Kaneda et al. (2003)]. Surprising results from this work include a strong bottleneck effect [Falkovich, 1994] near the dissipative subrange, and possibly a strong inertial range correction of about $k^{-0.1}$ to the usual $k^{-5/3}$ inertial range spectrum, so that the spectrum is $k^{-1.77}$. Similarly strong inertial range corrections have also been seen in simulations using a locally enhanced viscosity proportional to the modulus of the rate of strain matrix $|\mathbf{S}|^2$, which is also known as the Smagorinsky subgrid scale model. Here we also show the results of simulations with hyperviscosity, i.e. the $\nu \nabla^2$ diffusion operator has been replaced by a $\nu \lambda^6$ operator. Hyperviscosity greatly exaggerates the bottleneck effect, but it does not seem to affect the inertial range significantly; see Figure 4.

3 The bottleneck in turbulence

At large wavenumbers $k$, the energy spectrum

$$E(k) = \epsilon^{2/3} k^{5/3} f(k)$$  (1)
Figure 4: Comparison of energy spectra of the $4096^3$ meshpoints run \cite{Kaneda:2003} (solid line) and $512^3$ meshpoints runs with hyperviscosity (dash-dotted line) and Smagorinsky viscosity (dashed line). (In the hyperviscous simulation we use $\nu_3 = 5 \times 10^{-13}$.) The Taylor microscale Reynolds number of the Kaneda simulation is 1201, while the hyperviscous simulation of Ref. \cite{Haugen:2006} has an approximate Taylor microscale Reynolds number of $340 < \text{Re}_\lambda < 730$. For the Smagorinsky simulation the value of $\text{Re}_\lambda$ is slightly smaller.

is expected to have a viscous cutoff that is described by the function $f(k)$. However, $f(k)$ decreases not necessarily monotonically with $k$.

### 3.1 Navier–Stokes equation in Fourier space

For an incompressible fluid, the Fourier-transformed Navier–Stokes equation for $\hat{u}_k(t)$ can be written in the form

$$\frac{d}{dt} \hat{u}_k = -ik\hat{P}_k - \sum_{k=p+q} (\hat{u}_p \cdot iq) \hat{u}_q - \nu k^2 \hat{u}_k. \tag{2}$$

The pressure satisfies a Poisson-type equation, so

$$k^2 \hat{P}_k = ik \sum_{k=p+q} (\hat{u}_p \cdot iq) \hat{u}_q \tag{3}$$

and therefore

$$(ik\hat{P}_k)_i = -\frac{k_i k_j}{k^2} \sum_{k=p+q} (\hat{u}_p \cdot iq) \hat{u}_q \tag{4}$$

or

$$\frac{d}{dt} \hat{u}_k = -P(k) \sum_{k=p+q} (\hat{u}_p \cdot iq) \hat{u}_q - \nu k^2 \hat{u}_k, \tag{5}$$

where $P_{ij} = \delta_{ij} - k_i k_j/k^2$ is the projection operator, which projects out the non-solenoidal (irrotational) components.

An important point here is the fact that all nonlinear interactions proceed via triads in $k$ space where $k = p+q$. It turns out that viscosity suppresses those triads that reach deep into the dissipative subrange. This makes nonlinear energy transfer less efficient and can lead to a pileup of energy in the inertial range shortly before the dissipative subrange \cite{Falkovich:1994}. This is referred to as the bottleneck effect. To understand why it has not been a prominent effect in wind tunnel and atmospheric turbulence, we have to realize that most observed spectra have been obtained using hot-wire velocimetry and the Taylor hypothesis. We thus have to understand the relation between 1-D and 3-D energy spectra.
3.2 Relation between 1-D and 3-D energy spectra

To derive the relation between the three-dimensional spectrum $E(k)$ and the total one-dimensional spectrum $E_{1D}(k) \equiv E_1(k) + 2E_T(k)$, we consider a periodic box of volume $V = L_x L_y L_z$ with a turbulent velocity field $u(x)$, which has the Fourier transform

$$\hat{u}(k) = \frac{1}{\sqrt{(2\pi)^3}} \int_V e^{ik \cdot x} u(x) \, dx^3,$$

with the inversion

$$u(x) = \sqrt{\frac{V}{(2\pi)^3}} \int e^{-ik \cdot x} \hat{u}(k) \, dk^3. \tag{7}$$

The one-dimensional kinetic energy spectrum is

$$E_{1D}(k_z) = 2 \int \frac{|\langle \hat{u}(k) \rangle|^2}{2} \, dk_x \, dk_y, \quad (k_z \geq 0), \tag{8}$$

where $\langle \cdot \rangle$ denotes an ensemble average, and $k = (k_x, k_y, k_z)$. The factor 2 in Eq. (8) accounts for the fact that $E_{1D}$ does not distinguish between positive and negative $k_z$. Normalization of $E_{1D}(k_z)$ is such that

$$\int_0^\infty E_{1D}(k_z) \, dk_z = \frac{u_{\text{rms}}^2}{2} = \frac{1}{V} \int_V \frac{|\langle u(x) \rangle|^2}{2} \, dx^3. \tag{9}$$

Equation (8) can also be written as the $xy$-average

$$E_{1D}(k_z) = \frac{1}{L_x L_y} \int \langle |\hat{u}(x, y, k_z)|^2 \rangle \, dx \, dy \tag{10}$$

and is for homogeneous turbulence equal to $\langle |\hat{u}(x, y, k_z)|^2 \rangle$ at any point $(x, y)$.

The three-dimensional velocity energy spectrum is given by

$$E(k) = \int \frac{|\langle \hat{u}(k) \rangle|^2}{2} \, k^2 \, d\Omega_k, \tag{11}$$

where $d\Omega_k$ denotes the solid angle element in $k$-space. $E(k)$ satisfies the relation

$$\int_0^\infty E(k) \, dk = \frac{u_{\text{rms}}^2}{2}. \tag{12}$$

If $u$ is statistically isotropic in the sense that the ensemble average of the spectral energy of the velocity $\langle |u(k)|^2 \rangle$ is only a function of $k = |k|$, then $E(k)$ becomes

$$E(k) = 4\pi k^2 \frac{\langle |\hat{u}(k)|^2 \rangle}{2}. \tag{13}$$

To evaluate $E_{1D}$ in this case, we introduce cylindrical coordinates $(k_r, \phi, k_z)$ in $k$-space and write the double integral (8) in the form

$$E_{1D}(k_z) = 2 \int_0^\infty \frac{\langle |\hat{u}(k)|^2 \rangle}{2} 2\pi k_r \, dk_r \tag{14}$$

$$= 4\pi \int_{k_z}^\infty \frac{\langle |\hat{u}(k)|^2 \rangle}{2} k \, dk,$$

since $k^2 = k_r^2 + k_z^2$, and therefore $k_r^2 = k^2 - k_z^2$. Comparing with Eq. (13), we see that

$$E_{1D}(k_z) = \int_{k_z}^\infty \frac{E(k)}{k} \, dk, \tag{15}$$

the inversion of which gives

$$E(k) = -k \frac{dE_{1D}(k)}{dk} = -E_{1D} \frac{d \ln E_{1D}(k)}{d \ln k}. \tag{16}$$
4 Vorticity production and Karman–Howarth equation

In Handout 15, we have seen that turbulence is full of structures. It consists of vortex tubes, at least at small scales. They are not really produced by stretching, as one might have thought, but rather by rotational straining motion or shear. Vortices are thus modes similarly as you would make spaghetti.

4.1 Vorticity production

Mathematically, one can see that vorticity is being produced by the rate-of-strain matrix, \( s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \), because the vorticity equation

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u + \nu \nabla^2 \omega
\]

involves just the symmetric part (the antisymmetric part can be written as \( a_{ij} = -\frac{1}{2} \epsilon_{ijk} \omega_k \), so \( a_{ij} \omega_j = 0 \)). Thus, we have

\[
\frac{D\omega_i}{Dt} = s_{ij} \omega_j + \nu \nabla^2 \omega_i.
\]

It is therefore useful to study the principal axes of \( s_{ij} \) and to ask how the vorticity vector aligns itself in the strain field. Numerical simulations have shown conclusively that \( \omega \) aligns itself with the intermediate eigenvector of the rate-of-strain matrix (Vincent & Meneguzzi, 1991); see Figure 5 for a corresponding result in MHD, where it turns out that, by contrast, the magnetic field vector aligns itself with the direction of shear, which is at 45° angles with the directions of both stretching and compression (Brandenburg et al., 1995).

![Figure 5: Alignment of \( \omega \) and \( \mathbf{B} \) with the eigenvectors of the rate-of-strain tensor. Adapted from the Supplemental Material of [Brandenburg et al.](#) (2015).](image)

The production rate of enstrophy is given by

\[
\frac{d}{dt} \langle \frac{1}{2} \omega^2 \rangle = \langle s_{ij} \omega_i \omega_j \rangle - \nu \langle (\nabla \times \omega)^2 \rangle.
\]

(Incidently, a similar relation also holds for the magnetic field.) In a statistically steady state, \( \langle s_{ij} \omega_i \omega_j \rangle \) must be positive.

4.2 Skewness

The term \( \langle s_{ij} \omega_i \omega_j \rangle \) is cubic in the velocity derivatives, so it is not too surprising that one can also relate it to the cube of the velocity differences, \( (\Delta v)^3 \), at least for isotropic turbulence and in the limit of small separation \( r \); see Davidson (2015) for details. The calculation shows that

\[
\langle s_{ij} \omega_i \omega_j \rangle = -\frac{35}{2} \left[ \langle (\Delta v)^3 \rangle / r^3 \right]_{r \to 0}.
\]
The minus sign is here important. Many measurements of turbulence show that \( \langle (\Delta v)^3 \rangle \) is indeed negative. It means that the probability density function (PDF) of \( \Delta v \) is skewed to the left; see Figure 6 for an example of a PDF of \( \Delta v \) normalized by its rms value.

The shapes of PDFs is usually well characterized by its moments. The normalized third moment, \( \langle (\Delta v)^3 \rangle / \langle (\Delta v)^2 \rangle^{3/2} \), is called the skewness, and, independently of Reynolds number, its value is found to be in the range from \(-0.5\) to \(-0.4\). The fourth moment, \( \langle (\Delta v)^4 \rangle / \langle (\Delta v)^2 \rangle^2 \), is called the kurtosis. Its value would be 3 for a Gaussian distribution, but turbulence is highly non-Gaussian with elevated wings of the distribution, so the kurtosis is usually much bigger than 3.

The skewed distribution of velocity difference is thus crucial for the production and perhaps the very existence of vorticity! In fact, Kolmogorov found that it is related to the energy dissipation rate \( \epsilon \) via

\[
\langle (\Delta v)^3 \rangle = -\frac{4}{5} \epsilon r \quad \text{for } r \text{ within the inertial range.} 
\] (21)

where \( r \) is the separation between the two measurement points. This law is sometimes called the fourth-fifths law and it is an exact result. It follows as a special application of the Karman–Howarth equation, which will be explain next.

### 4.3 Karman–Howarth equation

The Karman–Howarth equation is a real-space equation for the two-point correlation function

\[
Q_{ij}(r) = \langle u_i(x)u_j(x+r) \rangle. 
\] (22)

It can also be written as \( Q_{ij}(x,x') = \langle u_i(x)u_j(x') \rangle \) and is therefore often abbreviated as \( \langle u_iu_j' \rangle \). The evolution equation for \( Q_{ij}(r) \) will have triple correlations on the right-hand side, which are defined as

\[
S_{ijk}(r) = \langle u_i(x)u_j(x)u_k(x+r) \rangle. 
\] (23)

Of particular interest for the following are special case such as

\[
Q_{xx}(r\hat{e}_x) = u^2 f(r), 
\]

\[
Q_{yy}(r\hat{e}_y) = u^2 g(r), 
\] (24)

which are also known as the longitudinal and lateral correlation functions. Likewise, we have

\[
S_{xxx}(r\hat{e}_x) = u^3 K(r). 
\] (25)

To derive the equation for \( Q_{ij} \), one starts with the momentum equation and multiplies by \( u' \), and likewise the momentum equation for \( u' \), which is then multiplied by \( u \). Thus,

\[
\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_k} (u_iu_k) - \frac{\partial}{\partial x_i} (p/\rho) + \nu \nabla_x^2 u_i, 
\] (27)
\[ \frac{\partial u'_j}{\partial t} = - \frac{\partial}{\partial x_k} (u'_j u'_k) - \frac{\partial}{\partial x_j} (\rho' / \rho) + \nu \nabla^2 u'_j. \]  (28)

Multiplying the first of these by \( u'_j \) and the second one by \( u_i \), and adding, we obtain
\[ \frac{\partial}{\partial t} (u'_i u'_j) = - \left< u'_i \frac{\partial}{\partial x'_k} (u'_j u'_k) \right> - \left< u'_i \frac{\partial}{\partial x'_k} (u_i u_k) \right> + \ldots \]  (29)

To continue from here, one has to use a number of tricks in the manipulation of two-point correlation tensors, such as

- Averaging and differentiation commute,
- \( \partial / \partial x_i \) and \( \partial / \partial x'_j \) can be replaced by \( -\partial / \partial r_i \) and \( \partial / \partial r_j \),
- \( \left< u_i u'_j u'_k \right> (r) = \left< u_j u_k u'_i \right> (-r) = - \left< u_j u_k u'_i \right> (r) \)

To see the latter, let us begin by noting that we can shift arbitrarily, so
\[ \left< u_i u'_j u'_k \right> (r) = \left< u_i (x) u_j (x + r) u_k (x) \right> = \left< u_i (x - r) u_j (x) u_k (x) \right> = \left< u'_i u_j u_k \right> (-r). \]  (30)

With this one obtains
\[ \frac{\partial}{\partial t} Q_{ij} = \frac{\partial}{\partial r_k} (S_{ikj} + S_{jki}) + 2\nu \nabla^2 Q_{ij}. \]  (31)

Of particular interest is the connection between \( f \) and \( K \), which turns out to be
\[ \frac{\partial}{\partial t} [u^2 f(r,t)] = \frac{1}{r^4} \frac{\partial}{\partial r} [r^4 u^3 K(r)] + 2\nu \frac{1}{r^4} \frac{\partial}{\partial r} [r^4 u^2 f'(r)] \]  (32)
where \( f'(r) \) is the \( r \) derivative. This is a special form of the Karman–Howarth equation. Applying it to turbulence leads to the following relation
\[ r^4 \frac{\partial}{\partial t} \left( \epsilon^{2/3} \epsilon^{2/3} \right) \sim r^4 (u/\ell)^{2/3} r^{2/3} \sim r^4 (r/\ell)^{2/3}. \]  (33)

**References**


