Lecture 8: Magnetic Fields

The evolution of the magnetic field is governed by Maxwell’s equations and by “material” equations, such as Ohm’s law. In Maxwell’s equations the Faraday displacement current is neglected, which is valid if either the relevant velocities in the problem are nonrelativistic (much smaller than the speed of light), or if the conductivity of the medium is so poor that there are no (ordinary) currents and only the displacement current balancing the $\nabla \times B$ term in Faraday’s equation. The latter leads to radio waves, for example. Both limiting cases are not considered in this course, although they may be of interest in some extreme astrophysical cases (near black holes or in the big bang).

1 The induction equation

The magnetic field, $B$, is governed by the equations

$$\frac{\partial B}{\partial t} + \nabla \times E = 0,$$

$$J = \nabla \times B/\mu_0,$$

$$E = -v \times B,$$

where $E$ is the electric field, $J$ the current density, $v$ the velocity, and $\mu_0$ the permeability. The field always satisfies $\nabla \cdot B$, but if this is satisfied by the initial condition, the evolution equation (1) will satisfy this automatically, because the divergence of a curl vanishes.

Substituting now $E$ from Equation (3) into (1) yields the induction equation

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B).$$

In many cases there is also resistivity, which can be important, but for now this will be ignored. However it is sometimes useful to rewrite the induction term using the identity

$$\nabla \times (v \times B) = -v \cdot B + B \cdot \nabla v + v \nabla \cdot B - B \nabla \cdot v.$$

Here, the term $v \nabla \cdot B$ vanishes, because $\nabla \cdot B$ vanishes. Thus, the induction equation can be written in the form

$$\frac{\partial B}{\partial t} + v \cdot B = B \cdot \nabla v - B \nabla \cdot v.$$

2 The momentum equation

The momentum equation is supplemented by an extra term, the Lorentz force $J \times B$, which then takes the form

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} J \times B + ..., $$

where the dots refer to other possible terms (friction, Coriolis force, etc.) The Lorentz force can also be written in the form

$$J \times B = \frac{1}{\mu_0} (B \cdot \nabla) B - \nabla \left( \frac{B^2}{2 \mu_0} \right).$$

The first term is commonly called the tension force, because it gives the derivative of the magnetic field in the direction of the magnetic field.
3 The magnetic energy equation

By dotting Equation (11) with $\mu_0^{-1}B$ we obtain

$$\frac{\partial}{\partial t} \left( \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} B \cdot \nabla \times E = 0.$$  

(9)

Noting that $\nabla \cdot (E \times B) = \epsilon_{ijk} \partial_i (E_j B_k)$ we can write, using the product rule,

$$\nabla \cdot (E \times B) = B \cdot \nabla \times E - E \cdot \nabla \times B.$$  

(10)

[Remember that $\epsilon_{ijk} = \epsilon_{jki} = -\epsilon_{jik}$] Using this relation we find that

$$\frac{\partial}{\partial t} \left( \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left( \frac{E \times B}{\mu_0} \right) + J \cdot E = 0.$$  

(11)

Writing $J \cdot E = J \cdot v \times B = \epsilon_{ijk} J_i v_j B_k$ we obtain the energy equation in the form

$$\frac{\partial}{\partial t} \left( \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left( \frac{E \times B}{\mu_0} \right) + v \cdot (J \times B) = 0.$$  

(12)

or, as an evolution equation for the magnetic energy density, $B^2/2\mu_0$, we have

$$\frac{\partial}{\partial t} \left( \frac{B^2}{2\mu_0} \right) = -\nabla \cdot \left( \frac{E \times B}{\mu_0} \right) - v \cdot (J \times B).$$  

(13)

The last term is important, because it gives the work done against the Lorentz force. If we write down the corresponding equation for the kinetic energy, that is obtained by dotting the momentum equation with $\rho v$, we obtain the term $+v \cdot (J \times B)$ on the rhs of that equation. In the equation for the sum of kinetic and magnetic energies this term then cancels. That means that this term describes the transfer of energy between kinetic and magnetic energy reservoirs. In many cases $v \cdot (J \times B)$ is negative, i.e. work is done against the Lorentz force and can lead to an increase of magnetic energy. This is the case in astrophysical dynamos. Again a hot research topic.

4 Hydromagnetic equations in conservative form

It is possible to write the magnetohydrodynamic (MHD) equations in conservative form, i.e. in the form

$$\frac{\partial}{\partial t} \text{(something)} + \nabla \cdot \text{(some corresponding flux)} = 0.$$  

(14)

In the case of the MHD equations the momentum and energy equations can be written in the form

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} \left[ \rho v_i v_j + \delta_{ij} p + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2) \right] = 0,$$  

(15)

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho e + \frac{1}{2\mu_0} B^2 \right) + \nabla \cdot \left[ v \left( \frac{1}{2} \rho v^2 + \rho e + p \right) + E \times B \right] = 0.$$  

(16)

Here we have used the continuity, momentum and energy equations together with the induction equation. The induction equation, $\partial B/\partial t = \nabla \times (u \times B)$, is readily written in conservative by just expanding the coefficients, so

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j B_i - u_i B_j) = 0,$$  

(17)

where the magnetic diffusivity is ignored.

In a sense those equations in conservative form are more fundamental than the momentum and energy equations that we have used so far, because if there are extra terms, for example in the continuity equation due to chemical reactions for example, this may yield additional terms in the momentum equation, but the equations in conservative form remain unchanged, unless those reactions lead to momentum injection.
5 One-dimensional MHD shocks

In one dimension the conservation equations can be integrated provided we are in the frame of reference comoving with the shock. We assume $B_x = \text{const}$ and $B_z = 0$, so we have

$$\rho v_x = J \quad (18)$$
$$\rho v_x^2 + p + \frac{1}{2\mu_0} (B_x^2 - B_y^2) = I \quad (19)$$
$$\frac{1}{2} v_x^2 + \frac{\gamma - 1}{\gamma - 1} \rho = E \quad (20)$$
$$v_x B_y = M \quad (21)$$

where $J$, $I$, $E$, and $M$ are integration constants. From this one can derive jump conditions very similar to those without fields. One ends up with a cubic equation (or a quartic equation in the more general case where a velocity component in the direction of the shock is allowed for). The different solutions correspond to different MHD wave types (slow and fast magnetosonic and Alfvén), but not always all those different types are possible, which is when the upstream velocity would be smaller than any of the other wave speeds. In the following we discuss those various wave types, although we won’t go into this very interesting field of MHD shocks to any greater detail.

6 Alfvén and magnetosonic waves

Here we demonstrate the technique of linearizing the MHD equations, which can then be used to study phenomena of small amplitude. This applies mainly to waves (provided they are of small amplitude), but it also applies to weak perturbations that can in some systems grow exponentially and would eventually no longer be small. In that case such an analysis can establish the possibility of an instability.

7 Linearizing the MHD equations

We now consider the isothermal MHD equations in the form

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = -c_s^2 \nabla \rho - \nabla \left( \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} (B \cdot \nabla) B. \quad (22)$$

Whenever we linearize, we have to linearize about something, which must be a solution to the equations. Those solutions may well be complicated, but then we won’t be able to do the analysis so easily on the blackboard. Therefore we now go for simple solutions. One solution is where the magnetic field is constant and uniform, i.e. $B = B_0$, and where the density is also constant, i.e. $\rho = \rho_0$, but the velocity is zero, i.e. $v = v_0 = 0$. In that case the terms on the lhs of Equation (22) vanish. The terms on the rhs of Equation (22) also vanish because because $\rho_0$ and $B_0$ are uniform. So we do have a solution now.

We now proceed by linearizing term by term, i.e. we write

$$\rho = \rho_0 + \rho', \quad (23)$$
$$v = v_0 + v' = v', \quad (24)$$
$$B = B_0 + B'. \quad (25)$$

The first term in (22) then becomes

$$\rho \frac{\partial v}{\partial t} = (\rho_0 + \rho') \frac{\partial v'}{\partial t} = \rho_0 \frac{\partial v'}{\partial t} + \rho' \frac{\partial v'}{\partial t}. \quad (26)$$

Here the second term is quadratic in the perturbations. Since those perturbations are small, then something small squared will be even smaller and will hence be neglected. The advection term is already nonlinear in the perturbations, because there is no zero-order term, so it vanishes altogether. Next, the pressure gradient term is linear already, so we have

$$c_s^2 \nabla \rho = c_s^2 \nabla \rho', \quad (27)$$

3
because \( \rho_0 \) is constant. For the magnetic pressure gradient we have

\[
\nabla \left( \frac{B^2}{2 \mu_0} \right) = \nabla \left( \frac{B_0 \cdot B'}{\mu_0} \right) + \text{quadratic terms,}
\]

(28)
because

\[
B^2 = (B_0 + B') \cdot (B_0 + B') = B_0^2 + 2B_0 \cdot B' + B'^2,
\]

(29)
of which only the second term survives. The first one is constant and gives no contribution under the gradient, and the last term is quadratic in the perturbations. Finally, the magnetic stretching term gives simply

\[
\frac{1}{\mu_0} (B \cdot \nabla) B = \frac{1}{\mu_0} (B_0 \cdot \nabla) B' + \text{quadratic terms.}
\]

(30)

Thus, our linearized momentum equation takes the form

\[
\rho_0 \frac{\partial v'}{\partial t} = -c_s^2 \nabla \rho' - \nabla \left( \frac{B_0 \cdot B'}{\mu_0} \right) + \frac{1}{\mu_0} (B_0 \cdot \nabla) B'.
\]

(31)

You can imagine that with a little bit of experience one can write down those equations straight away. We do this now with the remaining equations, the induction and continuity equations, looking at equations (6),

\[
\frac{\partial B'}{\partial t} = B_0 \cdot \nabla v' - B_0 \nabla \cdot v'.
\]

(32)

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot v' = 0,
\]

(33)

Next, we go through some examples of various degrees of interest.

8 Alfvén waves in the presence of a vertical magnetic field

We assume that all variables can be decomposed into plane waves in the form

\[
v_x' = \hat{v}_x e^{ik \cdot x - i\omega t},
\]

(34)

\[
v_y' = \hat{v}_y e^{ik \cdot x - i\omega t},
\]

(35)

\[
v_z' = \hat{v}_z e^{ik \cdot x - i\omega t},
\]

(36)

\[
B_x' = \hat{B}_x e^{ik \cdot x - i\omega t},
\]

(37)

\[
B_y' = \hat{B}_y e^{ik \cdot x - i\omega t},
\]

(38)

\[
B_z' = \hat{B}_z e^{ik \cdot x - i\omega t},
\]

(39)

\[
\rho' = \hat{\rho} e^{-ik \cdot x - i\omega t},
\]

(40)

which allows us to derive a set of algebraic equations. For the velocity components we have

\[
- i \omega \hat{v}_x = -ik_x \hat{\rho} \frac{c_s^2}{\rho_0} - ik_x \hat{B}_y \frac{B_0}{\mu_0 \rho_0} + ik_x \hat{B}_z \frac{B_0}{\mu_0 \rho_0}
\]

(41)

\[
- i \omega \hat{v}_y = -ik_y \hat{\rho} \frac{c_s^2}{\rho_0} - ik_y \hat{B}_z \frac{B_0}{\mu_0 \rho_0} + ik_y \hat{B}_y \frac{B_0}{\mu_0 \rho_0}
\]

(42)

\[
- i \omega \hat{v}_z = -ik_z \hat{\rho} \frac{c_s^2}{\rho_0}
\]

(43)

where the last two magnetic terms have canceled. The continuity equation is

\[
- i \omega \hat{\rho} = -ik_x \rho_0 \hat{u}_x - ik_y \rho_0 \hat{u}_y - ik_z \rho_0 \hat{u}_z
\]

(44)
The magnetic equations are now for the form

\[ -i\omega \hat{B}_x = +ik_z B_0 \hat{u}_x \]  \hspace{1cm} (45)
\[ -i\omega \hat{B}_y = +ik_z B_0 \hat{u}_y \]  \hspace{1cm} (46)
\[ -i\omega \hat{B}_z = -ik_z B_0 \hat{u}_x - ik_y B_0 \hat{u}_y \]  \hspace{1cm} (47)

where, again, the \( ik_z \) terms have canceled in the last equation.

It is now convenient to write those equations in matrix form

\[
\begin{pmatrix}
-i\omega & 0 & 0 & 0 & ik_x c_x^2 / \rho_0 & 0 & 0 & 0
-ik_z B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & -i\omega & 0 & ik_y c_y^2 / \rho_0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & -i\omega & 0 & 0
-ik_z B_0 & 0 & 0 & 0 & 0 & 0 & -i\omega & 0
ik_x B_0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega
\end{pmatrix}
\begin{pmatrix}
\hat{v}_x \\
\hat{v}_y \\
\hat{v}_z \\
\hat{p} \\
\hat{B}_x \\
\hat{B}_y \\
\hat{B}_z
\end{pmatrix} = 0. \hspace{1cm} (48)
\]

We won’t consider this rather large system of equations at this point. Instead we want to observe what happens when we assume that the system is one-dimensional. Hence, we ignore the \( x \) and \( y \) components of the wave vector, so our matrix becomes

\[
\begin{pmatrix}
-i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & -i\omega & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & -i\omega & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & -i\omega & 0 & 0 & 0 & 0
-ik_z B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & -ik_z B_0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega
\end{pmatrix}
\begin{pmatrix}
\hat{v}_x \\
\hat{v}_y \\
\hat{v}_z \\
\hat{p} \\
\hat{B}_x \\
\hat{B}_y \\
\hat{B}_z
\end{pmatrix} = 0. \hspace{1cm} (49)
\]

We notice that the last equation disappears altogether. We notice further that the third and fourth equations decouple from the remaining four equations. In order to have non-trivial solutions (where the hatted variables themselves don’t vanish), we have to require that the determinant of the matrix vanishes. Because the matrix decouples into two matrices, \( M_1 \) and \( M_2 \), we have to require that either \( \det M_1 = 0 \) or \( \det M_2 = 0 \), where

\[
M_1 = \begin{pmatrix}
-i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\omega & 0 & 0 & 0 & 0 & 0 \\
-ik_z B_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -ik_z B_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega
\end{pmatrix}, \hspace{1cm} M_2 = \begin{pmatrix}
-i\omega & ik_z c_z^2 / \rho_0 \\
0 & -i\omega \\
-ik_z B_0 & 0 \\
0 & -ik_z B_0
\end{pmatrix} \hspace{1cm} (50)
\]

There are two separate dispersion relations. From \( \det M_2 = 0 \) we have

\[
\omega^2 = c_z^2 k_z^2. \hspace{1cm} (51)
\]

The resulting dispersion relation from \( \det M_2 = 0 \) is

\[
\omega^4 - 2v_A^2 k^2 \omega^2 + v_A^4 k^4 = 0. \hspace{1cm} (52)
\]

This is a biquadratic equation, or a quadratic equation in \( \omega^2 \). However, in this case it can also be written as

\[
(\omega^2 - v_A^2 k^2)^2 = 0, \hspace{1cm} (53)
\]

which simply means that there are Alfvén waves were either the \( x \) or the \( y \) components of the field are involved.
9 One-dimensional Alfvén waves revisited

Given that in one dimension the Alfvén waves simplify significantly we begin all over again, focusing attention immediately on the essential points. The linearized, pressureless, ideal MHD equations in one dimension, in the presence of a vertical magnetic field, are

\[
\frac{\partial v'_x}{\partial t} = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_x}{\partial z},
\]

\[
\frac{\partial B'_x}{\partial t} = \frac{\partial v'_x}{\partial z},
\]

Differentiating the first equation in time and inserting the second equation for \(\partial B'_x/\partial t\) yields a wave equation, similar to the equation describing sound waves. In the more complicated situations described below it turns out to be easier however to take the solution to be in the form

\[
v'_x = \hat{v}_xe^{ikz-i\omega t},
\]

\[
B'_x = \hat{B}_xe^{ikz-i\omega t},
\]

which allows us to derive a set of two algebraic equations,

\[
-\omega \hat{v}_x - 2\Omega \hat{v}_y = \frac{B_0}{\rho_0 \mu_0} \hat{B}_x,
\]

\[
-\omega \hat{B}_x = B_0 \hat{v}_x.
\]

It is convenient to write those equations in matrix form

\[
\begin{pmatrix}
-\omega & -ik \frac{B_0}{\rho_0 \mu_0} \\
-ikB_0 & -\omega
\end{pmatrix}
\begin{pmatrix}
\hat{v}_x \\
\hat{B}_x
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

For a nontrivial solution the determinant of the governing matrix has to vanish

\[
\det M = \det \left(\begin{pmatrix}
-\omega & -ik \frac{B_0}{\rho_0 \mu_0} \\
-ikB_0 & -\omega
\end{pmatrix}\right) = -\omega^2 + \frac{B_0^2}{\mu_0 \rho_0} k^2 = 0,
\]

This leads to the dispersion relation \(\omega = \omega(k)\) with

\[
\omega = \pm v_A k,
\]

where \(v_A\) is the Alfvén speed with \(v_A^2 = B_0^2/(\rho_0 \mu_0)\).

Unlike sound waves, where the restoring force is the pressure gradient, for Alfvén waves the restoring force is the magnetic field. Another important difference is the fact that now the field has only components perpendicular to the direction of propagation. Here the direction of propagation is \(z\), but \(B\) has only a nonvanishing component in the \(x\) direction. For sound waves, on the other hand, the velocity has only a component in the direction of propagation.

9.1 The effects of rotation

We now assume that we are in a rotating system of reference, where the rotation axis is the \(z\) axis, which is also the direction of the applied magnetic field, so \(\Omega = (0, 0, \Omega)\) and so the Coriolis force (per unit mass) is

\[
F_{Cor} = -2\Omega \times v = \begin{pmatrix}
0 \\
0 \\
-2\Omega
\end{pmatrix} \times \begin{pmatrix}
v_x \\
v_y \\
v_z
\end{pmatrix} = \begin{pmatrix}
+2\Omega v_y \\
-2\Omega v_x \\
0
\end{pmatrix}.
\]

Adding this term to the equations causes immediately some coupling to the \(y\) components of both the velocity and the magnetic field. The linearized, pressureless, ideal MHD equations in one dimension, in the presence of rotation, are then

\[
\frac{\partial v'_x}{\partial t} - 2\Omega v'_y = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_x}{\partial z},
\]

\[
\frac{\partial B'_x}{\partial t} = \frac{\partial v'_x}{\partial z},
\]
\[ \frac{\partial v'_y}{\partial t} + 2\Omega v'_x = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B'_y}{\partial z}, \]
\[ \frac{\partial B'_x}{\partial t} = B_0 \frac{\partial v'_x}{\partial z}, \]
\[ \frac{\partial B'_y}{\partial t} = B_0 \frac{\partial v'_y}{\partial z}. \]

As usual, we assume the solution to be of the form
\[ v'_x = \hat{v}_x e^{ikz - i\omega t}, \]
\[ v'_y = \hat{v}_y e^{ikz - i\omega t}, \]
\[ B'_x = \hat{B}_x e^{ikz - i\omega t}, \]
\[ B'_y = \hat{B}_y e^{ikz - i\omega t}, \]

which allows us to derive a set of algebraic equations.
\[ -i\omega \hat{v}_x - 2\Omega \hat{v}_y = \frac{B_0}{\rho_0 \mu_0} i k \hat{B}_x, \]
\[ -i\omega \hat{v}_y + 2\Omega \hat{v}_x = \frac{B_0}{\rho_0 \mu_0} i k \hat{B}_y, \]
\[ -i\omega \hat{B}_x = B_0 i k \hat{v}_x, \]
\[ -i\omega \hat{B}_y = B_0 i k \hat{v}_y. \]

It is convenient to write those equations in matrix form
\[ \begin{pmatrix} -i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\ 2\Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & -i\omega & 0 \\ 0 & -ik B_0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{B}_x \\ \hat{B}_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \]

For a nontrivial solution the determinant of the governing matrix has to vanish
\[ \det M \equiv \det \begin{pmatrix} -i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\ 2\Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & -i\omega & 0 \\ 0 & -ik B_0 & 0 & -i\omega \end{pmatrix}. \]

To calculate the determinant we split it into sub-determinants:
\[ \det M = -i\omega \det \begin{pmatrix} -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ 0 & -i\omega & 0 \\ -ik B_0 & 0 & -i\omega \end{pmatrix} + 2\Omega \det \begin{pmatrix} 2\Omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} - ik \frac{B_0}{\mu_0 \rho_0} \det \begin{pmatrix} 2\Omega & -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\ -ik B_0 & 0 & 0 \\ 0 & -ik B_0 & -i\omega \end{pmatrix}. \]

Determinants of 3 x 3 matrices are easy to calculate, so
\[ \det M = -i\omega \left[ (-i\omega)^3 - (-i\omega) \left( -k^2 \frac{B_0^2}{\mu_0 \rho_0} \right) \right] + 2\Omega \left[ 2\Omega (-i\omega)^2 \right] - ik \frac{B_0}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} - (-i\omega)^2 (\hat{B}_0 B_0) \right]. \]

Thus,
\[ \det M = -\omega^2 \left[ -\omega^2 + k^2 \frac{B_0^2}{\mu_0 \rho_0} \right] - 4\Omega^2 \omega^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} + \omega^2 \right], \]
or
\[
\det M = \omega^4 - \omega^2 k^2 v_A^2 - 4\Omega^2 \omega^2 + (k^2 v_A^2)^2 - k^2 v_A^2 \omega^2,
\] (81)
where \( v_A^2 = B_0^2/\mu_0 \rho_0 \) is the square of the Alfvén speed. Setting the determinant to zero leads to the dispersion relation
\[
\omega^4 - \omega^2 (2v_A^2 k^2 + 4\Omega^2) + v_A^4 k^4 = 0.
\] (82)
This is a biquadratic equation, or a quadratic equation in \( \omega^2 \); see Figure 1.

![Figure 1: Dispersion relation for slow magnetosonic and Alfvén waves with rotation and no shear. Blue solid lines denote slow magnetosonic waves and red dashed lines Alfvén waves. Only the part with \( k^2 > 0 \) is meant to be meaningful. (Negative \( k^2 \) correspond to evanescant waves.)](image)

9.2 The effects of rotation and shear

The linearized, pressureless, ideal MHD equations in one dimension, in the presence of rotation and shear with a vertical magnetic field, are
\[
\frac{\partial v_x'}{\partial t} - 2\Omega v_y' = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B_x'}{\partial z},
\] (83)
\[
\frac{\partial v_y'}{\partial t} + \frac{1}{2}\Omega v_x' = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B_y'}{\partial z},
\] (84)
\[
\frac{\partial B_x'}{\partial t} = B_0 \frac{\partial v_x'}{\partial z},
\] (85)
\[
\frac{\partial B_y'}{\partial t} = B_0 \frac{\partial v_y'}{\partial z} - \frac{3}{2}\Omega B_x',
\] (86)

We assume the solution to be of the form
\[
v_x' = \hat{v}_x e^{ikz - i\omega t},
\] (87)
\[
v_y' = \hat{v}_y e^{ikz - i\omega t},
\] (88)
\[
B_x' = \hat{B}_x e^{ikz - i\omega t},
\] (89)
\[
B_y' = \hat{B}_y e^{ikz - i\omega t},
\] (90)
which allows us to derive a set of algebraic equations.
\[
-i\omega \hat{v}_x - 2\Omega \hat{v}_y = \frac{B_0}{\rho_0 \mu_0} i k \hat{B}_x,
\] (91)
\[
-i\omega \hat{v}_y + \frac{1}{2}\Omega \hat{v}_x = \frac{B_0}{\rho_0 \mu_0} i k \hat{B}_y,
\] (92)
\[
-i\omega \hat{B}_x = B_0 i k \hat{v}_x,
\] (93)
For a nontrivial solution the determinant of the governing matrix has to vanish.

\[ -i\omega \tilde{B}_y = B_0 i k \hat{v}_y - \frac{3}{2} \Omega \tilde{B}_x. \]  

(94)

It is convenient to write those equations in matrix form

\[
\begin{pmatrix}
-i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\
\frac{1}{2} \Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\
-ik B_0 & 0 & -i\omega & 0 \\
0 & -ik B_0 & \frac{3}{2} \Omega & -i\omega \\
\end{pmatrix}
\begin{pmatrix}
\hat{v}_x \\
\hat{v}_y \\
\tilde{B}_x \\
\tilde{B}_y \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}.
\]  

(95)

For a nontrivial solution the determinant of the governing matrix has to vanish

\[ \det M \equiv \det \begin{pmatrix}
-i\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\
\frac{1}{2} \Omega & -i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\
-ik B_0 & 0 & -i\omega & 0 \\
0 & -ik B_0 & \frac{3}{2} \Omega & -i\omega \\
\end{pmatrix}. \]  

(96)

To calculate the determinant we split it into sub-determinants:

\[ \det M = -i\omega \det \begin{pmatrix}
-i\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\
0 & -i\omega & 0 \\
-ik B_0 & \frac{3}{2} \Omega & -i\omega \\
\end{pmatrix} + 2\Omega \det \begin{pmatrix}
\frac{1}{2} \Omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\
-ik B_0 & -i\omega & 0 \\
0 & \frac{3}{2} \Omega & -i\omega \\
\end{pmatrix} - ik \frac{B_0}{\mu_0 \rho_0} \det \begin{pmatrix}
\frac{1}{2} \Omega & -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\
-ik B_0 & 0 & -ik B_0 \\
0 & -i\omega & -i\omega \\
\end{pmatrix}. \]  

(97)

Determinants of 3 × 3 matrices are easy to calculate, so

\[ \det M = -i\omega \left[ (-i\omega)^3 - (-i\omega) \left( -k^2 \frac{B_0^2}{\mu_0 \rho_0} \right) \right] + 2\Omega \left[ \frac{1}{2} \Omega (-i\omega)^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3}{2} \Omega \right] - ik \frac{B_0}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} (-ik B_0) - (-i\omega)^2 (-ik B_0) \right]. \]  

(98)

Thus,

\[ \det M = -\omega^2 \left[ -\omega^2 + k^2 \frac{B_0^2}{\mu_0 \rho_0} \right] + 2\Omega \left[ -\frac{1}{2} \Omega \omega^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3}{2} \Omega \right] - k^2 \frac{B_0^2}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} + \omega^2 \right], \]  

(99)

or

\[ \det M = \omega^4 - \omega^2 k^2 v_A^2 + \Omega^2 \omega^2 - 3\Omega^2 k^2 v_A^2 + (k^2 v_A^2)^2 - k^2 v_A^2 \omega^2, \]  

(100)

where \( v_A^2 = B_0^2 / (\mu_0 \rho_0) \) is the square of the Alfvén speed. Setting the determinant to zero leads to the dispersion relation

\[ \omega^4 - \omega^2 (2v_A^2 k^2 + \Omega^2) + v_A^2 k^2 (v_A^2 k^2 - 3\Omega^2) = 0. \]  

(101)

This is a biquadratic equation, or a quadratic equation in \( \omega^2 \). There are two solutions for \( \omega^2, \omega_1^2 \) and \( \omega_2^2 \), say. We can therefore write the dispersion relation in the form

\[ (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = 0, \]  

(102)

or

\[ \omega^4 - \omega^2 (\omega_1^2 + \omega_2^2) + \omega_1^2 \omega_2^2 = 0. \]  

(103)

Comparing with Equation 126 we see that one of the two solutions is negative, i.e. \( \omega_1^2 \omega_2^2 < 0 \) when

\[ k^2 v_A^2 < 3\Omega^2. \]  

(104)

In that case \( \omega \) is plus or minus an imaginary number. Since the solution is proportional to \( e^{-i\omega t} \) that means that the solution behaves like

\[ e^{-i\omega t} = e^{\pm |\text{Im}(\omega)| t}. \]  

(105)
The solution for which the upper sign applies grows fastest. We thus say that the solution is unstable. Eventually the magnitude of the velocity and magnetic field perturbations will be so large that the linearized equations are no longer valid.

Note that we were able to draw conclusions about stability and instability even though we did not actually solve the dispersion relation. This happens quite often, especially with higher order polynomials. But in the present case an explicitly solution can easily be written down. It is

\[ \omega_{1,2}^2 = v_A^2 k^2 + \frac{1}{2} \Omega^2 \pm \sqrt{4v_A^2 k^2 \Omega^2 + \frac{1}{4} \Omega^4} \]  

(106)

We see that the presence of shear has a destabilizing effect on the slow magnetosonic waves. This instability is also known as magneto-rotational (or sometimes Balbus-Hawley) instability and it has great relevance for causing turbulence in accretion discs. The term magneto-rotational instability is frequently abbreviated by MRI. The maximum growth rate is reached when \( v_A^2 k^2 = \frac{15}{16} \Omega^2 \), and the corresponding value of \( \omega^2 \) is then \( -\omega^2 = \frac{9}{16} \Omega^2 \approx 0.56 \Omega^2 \), or \( \text{Im} \omega = \frac{3}{4} \Omega \).

Figure 2: Dispersion relation for slow magnetosonic and Alfven waves. Solid lines denote Alfven waves. For them \( \omega^2 \) is always positive. Dashed lines refer to slow magnetosonic waves. For not too large values of \( k \) they can become unstable (\( \omega^2 < 0 \), so \( \omega \) becomes imaginary!).

### 9.3 Eigenfunction for the MRI

A simple way to get the eigenfunction of the MRI, of in fact of any linear eigenvalue problem, is to use the original equations starting with the most simple one. The simplest one is Equation (93), because it involves just 2 terms. We may choose the coefficient in front of \( \hat{v}_x \) to one, so

\[ \hat{v}_x = 1 \]  

(107)

Using Equation (93) it follows that

\[ \hat{B}_x = -\frac{i k}{i \omega} B_0 \]  

(108)

Next we use Equation (91) to find \( \hat{v}_y \)

\[ \hat{v}_y = -\frac{i \omega + \frac{B_0}{\rho_0 \mu_0} i k \left( -\frac{i k}{i \omega} B_0 \right)}{2 \Omega} \]  

(109)

or

\[ \hat{v}_y = \frac{\omega^2 - v_A^2 k^2}{2 \Omega i \omega} \]  

(110)
Finally, we need to calculate $\hat{B}_y$, but we have two equations still unused, \((92)\) and \((94)\). We use first Equation \((92)\) to obtain
\[
\hat{B}_y = -\frac{i\omega \hat{v}_y + \frac{1}{2} \Omega \hat{v}_x}{ikv_A^2} B_0
\] (111)
or, using the above results,
\[
\hat{B}_y = -\frac{\omega^2 - v_A^2 k^2}{2\Omega k^2 v_A^2} B_0
\] (112)
so
\[
\hat{B}_y = ik B_0 \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2}
\] (113)
Finally, we can use the last equation, \((94)\) to check everything, so we plug in the results obtained so far.
We have
\[
-\omega \hat{B}_y = B_0 i k \hat{v}_y - \frac{3}{2} \Omega \hat{B}_x.
\] (114)
so
\[
-\omega (ik B_0) \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2} = B_0 i k \frac{\omega^2 - v_A^2 k^2}{2\Omega i \omega} - \frac{3}{2} \Omega \left( -\frac{i k}{\omega} B_0 \right).
\] (115)
We cancel $ik B_0$ on both sides, and multiply by $i \omega$, so
\[
\omega^2 \frac{\omega^2 - v_A^2 k^2 - \Omega^2}{2\Omega k^2 v_A^2} = \omega^2 - v_A^2 k^2 + \frac{3}{2} \Omega.
\] (116)
Multiplying by $2\Omega k^2 v_A^2$ yields
\[
\omega^2 (\omega^2 - v_A^2 k^2 - \Omega^2) = (\omega^2 - v_A^2 k^2) k^2 v_A^2 + 3 \Omega^2 k^2 v_A^2.
\] (117)
which gives the old biquadratic equation,
\[
\omega^4 - \omega^2 (2v_A^2 k^2 - \Omega^2) + (k^2 v_A^2 - 3 \Omega^2) k^2 v_A^2 = 0,
\] (118)
so everything seems to have gone alright. Anyway, the eigenfunction is therefore
\[
\begin{pmatrix}
\hat{v}_x \\
\hat{v}_y \\
\hat{B}_x \\
\hat{B}_y
\end{pmatrix} =
\begin{pmatrix}
\omega^2 - v_A^2 k^2 \\
\omega^2 - v_A^2 k^2 - \Omega^2
\end{pmatrix}
\] (119)

### 9.4 The effect of magnetic and ambipolar diffusion

For a nontrivial solution the determinant of the governing matrix has to vanish
\[
\det M \equiv \det \left( \begin{array}{cccc}
-\omega & -2\Omega & -ik \frac{B_0}{\mu_0 \rho_0} & 0 \\
\frac{1}{2} \Omega & -\omega & 0 & -ik \frac{B_0}{\mu_0 \rho_0} \\
-ik B_0 & 0 & -i\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\
0 & -ik B_0 & \frac{3}{2} \Omega & -i\omega
\end{array} \right).
\] (120)
where $-i\omega = -i \omega - \eta k^2$. To calculate the determinant we split it into sub-determinants:
\[
\det M = -i \omega \det \left( \begin{array}{cc}
-\omega & -ik \frac{B_0}{\mu_0 \rho_0} \\
0 & -i\omega
\end{array} \right) + 2\Omega \det \left( \begin{array}{cc}
\frac{1}{2} \Omega & 0 \\
-ik B_0 & -i\omega
\end{array} \right) + \det \left( \begin{array}{cc}
-ik B_0 & \frac{1}{2} \Omega \\
0 & -ik B_0
\end{array} \right).
\] (121)
Determinants of $3 \times 3$ matrices are easy to calculate, so

\[
\det M = -i\omega \left[ (-i\omega)(-i\omega)^2 - (-i\omega) \left( -k^2 \frac{B_0^2}{\mu_0 \rho_0} \right) \right] + 2\Omega \left[ \frac{1}{2} \Omega(-i\omega)^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3\Omega}{2} \right]
\]

\[
-ik \frac{B_0}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} (-ikB_0) - (-i\omega)(-i\omega)(-ikB_0) \right].
\]

(122)

\[
\det M = -i\omega \left[ (-i\omega)(-i\omega)^2 - (-i\omega) \left( -k^2 \frac{B_0^2}{\mu_0 \rho_0} \right) \right] + 2\Omega \left[ \frac{1}{2} \Omega(-i\omega)^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3\Omega}{2} \right]
\]

\[
-ik \frac{B_0}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} (-ikB_0) - (-i\omega)(-i\omega)(-ikB_0) \right].
\]

(123)

Thus,

\[
\det M = -\omega^2 \left[ -\omega^2 + \sigma k^2 \frac{B_0^2}{\mu_0 \rho_0} \right] + 2\Omega \left[ \frac{1}{2} \Omega\omega^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0} \frac{3\Omega}{2} \right] - k^2 \frac{B_0^2}{\mu_0 \rho_0} \left[ -k^2 \frac{B_0^2}{\mu_0 \rho_0} + \sigma \omega^2 \right],
\]

(124)

where $\sigma = \frac{\omega}{\omega}$ has been introduced as a short hand. This can be simplified to give

\[
\det M = \omega^4 - \sigma \omega^2 k^2 v_A^2 - \Omega^2 \omega^2 - 3\Omega^2 k^2 v_A^2 + (k^2 v_A^2)^2 - k^2 v_A^2 \sigma \omega^2,
\]

(125)

where $v_A^2 = B_0^2/(\mu_0 \rho_0)$ is the square of the Alfvén speed. Setting the determinant to zero leads to the dispersion relation

\[
\omega^4 - \omega^2 (2\sigma v_A^2 k^2 + \Omega^2) + v_A^2 k^2 (v_A^2 k^2 - 3\Omega^2) = 0.
\]

(126)

9.5 Alternative formulation

Ignore magnetic pressure gradient (assume that it is being balanced by the gas pressure gradient), so we have

\[
B_0 \cdot \nabla b = B_0 \partial_y b = B_0 \begin{pmatrix} \partial_y & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_y \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}
\]

(127)

\[
B_0 \cdot \nabla u = B_0 \begin{pmatrix} \partial_y & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_y \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}
\]

(128)

so the full matrix is

\[
\hat{A} \equiv -i \hat{L}(t) = \begin{pmatrix} 0 & 2\Omega & 0 & ik_y & 0 & 0 \\ -(2-q)\Omega & 0 & 0 & 0 & ik_y & 0 \\ 0 & 0 & 0 & 0 & 0 & ik_y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ ik_y & 0 & 0 & 0 & 0 & 0 \\ 0 & ik_y & 0 & -q\Omega & 0 & 0 \\ 0 & 0 & ik_y & 0 & 0 & 0 \end{pmatrix}
\]

(129)

Corresponds to the dispersion relation

\[
-\omega^6 + \omega^4 \left[ 3v_A^2 k^2 + 2(2-q)\Omega^2 \right] - \omega^2 v_A^2 k^2 \left[ 3v_A^2 k^2 + 4(1-q)\Omega^2 \right] + v_A^4 k^4 (v_A^2 k^2 - 2q\Omega^2) = 0.
\]

(130)

Has unstable eigenvalues for $3\Omega^2 > v_A^2 k^2$, just like in the axisymmetric case.

10 Magnetosonic waves

We now consider the linearized MHD equations in vector notation,

\[
-i \omega \hat{v} = -ik \frac{c_s^2}{\rho_0} \hat{\rho} - ik \frac{B_0 \cdot \hat{B}}{\mu_0 \rho_0} + \frac{ik \cdot B_0}{\mu_0 \rho_0} \hat{B},
\]

(131)
the special case
Here we only bother to give one particular eigenfunction for the case where $\omega = 0$, and of course the values of the sound speed and the Alfvén speed. The dependence of the frequency $\omega$ on $k$ now becomes

$$
- \omega^2 \mathbf{v} = -i k \cdot \mathbf{v},
$$

$$
- i \omega \mathbf{B} = (ik \cdot B_0) \hat{\mathbf{v}} - (ik \cdot \hat{\mathbf{v}}) B_0.
$$

We multiply Equation (131) by $-i$ and substitute for $-ik \cdot \mathbf{v}$ and $-i \omega \mathbf{B}$ using Eqs. (132) and (133), so

$$
- \omega^2 \hat{\mathbf{v}} = -i k \left( \frac{c_s^2}{\rho_0} \right) (-i k \cdot \hat{\mathbf{v}}) - \frac{i k}{\mu_0 \rho_0} \left[ (ik \cdot B_0)(\hat{\mathbf{v}} \cdot B_0) - (ik \cdot \hat{\mathbf{v}}) B_0^2 \right] + \frac{i k}{\mu_0 \rho_0} \left[ (ik \cdot B_0)\hat{\mathbf{v}} - (ik \cdot \hat{\mathbf{v}}) B_0 \right],
$$

or

$$
(k^2 v_A^2 - \omega^2) \mathbf{v} = k \left[ -(k \cdot \hat{\mathbf{v}}) (c_s^2 + v_A^2) + (k \cdot B_0)(\hat{\mathbf{v}} \cdot B_0) v_A^2 \right] + B_0 (k \cdot B_0) (k \cdot \hat{\mathbf{v}}) v_A^2.
$$

We now assume that the wave vector points in the $x$ direction and that the magnetic field vector lies in the $x-y$ plane, i.e.

$$
k = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} B_0.
$$

Therefore,

$$
\mathbf{v} \cdot B_0 = B_0 (\hat{v}_x \cos \psi + \hat{v}_y \sin \psi), \quad k \cdot B_0 = k B_0 \cos \psi, \quad k \cdot \mathbf{v} = k \hat{v}_x.
$$

With this the $z$ component of the right hand side vanishes, but because $\hat{v}_z$ does not generally vanish we have $k^2 v_A^2 - \omega^2 = 0$, which gives already two possible solutions:

$$
\omega = \pm k v_A.
$$

For the remaining two components we have to satisfy the $x$ and $y$ components separately, i.e.

$$
(k^2 v_A^2 - \omega^2) \hat{v}_x = k \left[ -\hat{v}_x (c_s^2 + v_A^2) + \cos \psi (\hat{v}_x \cos \psi + \hat{v}_y \sin \psi) v_A^2 \right] + \cos \psi (\hat{v}_x \cos \psi) \hat{v}_x k^2 v_A^2.
$$

$$
(k^2 v_A^2 - \omega^2) \hat{v}_y = \sin \psi \cos \psi v_A^2 k^2 \hat{v}_x
$$

This is a system of two algebraic equations. Written in matrix form we have,

$$
\begin{pmatrix}
(k^2 v_A^2 - \omega^2) + (c_s^2 + v_A^2) k^2 - 2 \cos^2 \psi v_A^2 k^2 \\
- \cos \psi \sin \psi k v_A^2 k^2 \\
\end{pmatrix}
\begin{pmatrix}
\hat{v}_x \\
\hat{v}_y \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
$$

which leads to the dispersion relation

$$
[(k^2 v_A^2 - \omega^2) + (c_s^2 + v_A^2) k^2 - 2 \cos^2 \psi v_A^2 k^2] (k^2 v_A^2 - \omega^2) + \cos^2 \psi \sin^2 \psi v_A^4 k^4 = 0
$$

We note that there are altogether three wave types, slow and fast magnetosonic waves and Alfvén waves. The frequency depends on the values of $k$, the angle between the wave vector $k$ and the magnetic field $B_0$, and of course the values of the sound speed and the Alfvén speed. The dependence of the frequency as a function of the angle between $k$ and $B_0$ is shown in Figure 3 for different values of $c_s$ and $v_A$. In the special case $\psi = 0$ we have

$$
(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2) = 0.
$$

### 10.1 Eigenfunctions

Here we only bother to give one particular eigenfunction for the case where $B_0$ and $k$ are aligned. In that case, Alfvén waves and fast magnetosonic waves are degenerate (see Figure 3), so we are left with two modes.

$$
v_x = \epsilon_c c_s \sin k(x - c_s t), \quad \ln \rho = \epsilon_c k(x - c_s t),
$$

$$
v_y = \epsilon_A^2 v_A \sin k(x - v_A t), \quad b_y = \epsilon_A^2 B_0 \sin k(x - v_A t).
$$

We note that the vector potential is $A = (0, 0, A_z)$ with $A_z = \epsilon_A k^{-1} B_0 \sin k(x - v_A t)$. 
Figure 3: Dispersion relation for Alfvén waves. Solid lines denote the fast magneto-sonic waves, dashed lines Alfvén waves, and dotted lines slow magneto-sonic waves.