Lecture 6: Thin accretion disks

We have already mentioned the formation of discs in connection with the planetary system. The same idea applies also to binary stars and to centres of galaxies. In the former case mass is being transferred from one component to the other either via a wind (in the case of a hot donor) or via Roche lobe overflow.

In the second case, the centres of galaxies, the situation is even more reminiscent of planetary systems, but on a larger scale and with different temperatures involved. Both in the centres of galaxies as well as in protostellar discs one starts from a more-or-less homogeneous cloud that collapses, but because of rotation there will be a time during the collapse when the centrifugal force begins to balance gravity. This is the point when a disc is formed and further radial infall is only possible because there is either a viscous or a magnetic torque that removes angular momentum outwards.

In the following we consider the standard case of a viscous accretion disc with given viscosity. Because discs are thin it is possible to calculate the radial structure analytically. This thin-disc solution is due to Shakura & Sunyaev (1973) and it has become an important milestone in modern astrophysics.

Before deriving the governing equations and the standard solution we first explain in qualitative terms how discs occur in binary systems.

1 Binary stars

More than 50% of the stars are binary stars. Some say the higher probability of finding binary stars is because they are counted twice. Anyway, binary stars can be divided into three different classes: detached, semi-detached, and contact binaries.

Detached binaries are relatively boring from a hydrodynamical view point, unless they are close enough to lead to tidal coupling between the two components, for example. However, detached binaries may evolve. The heavier of the two components will at some point develop into a red giant, which is so big that it will “fill its Roche-lobe”. That means matter will start to flow to the other component. What is now crucial for us is the fact that the overflowing matter has angular momentum and will therefore be unable to fall onto the secondary star directly. This is why a disc is formed. We encountered a similar problem earlier in connection with star formation where, again, matter was unable to fall directly onto the central object because of rotation. Instead a disc was formed. Such discs are typically viscous in the sense that turbulence in them leads to friction which allows angular momentum to be removed, so matter can move in further towards the central object.

2 Accretion discs

In this section we explain the basic theory used to describe the radial structure of accretion discs. This theory is mathematically simple, because it involves only algebraic equations, and it is physically rather powerful, because complicated physical processes can be taken into account.

As we mentioned in the beginning, the theory is important because it is applicable to a wide range of different astrophysical bodies: protostellar and protoplanetary discs, discs in X-ray binaries and other binary systems, and discs in active galactic nuclei and centres of galaxies.

To obtain the governing equations we derive vertically integrated equations for the mechanical equilibrium in the vertical and horizontal directions, as well as the thermal equilibrium. An important quantity in this business is the vertically integrated density $\Sigma \equiv \int_{-\infty}^{\infty} \rho \, dz$, where $H$ is the half thickness of the disc. However, no attempt is usually made to consider the vertical integration as an exact procedure. It is merely just a replacement of $\rho$ by its value in the central plane, $\rho_c$, which is then replaced in all the equations by $\Sigma/H$. 
2.1 Vertical equilibrium

In Section ?? we showed that the vertical component of gravity in a disc is

\[ g_z = -\Omega^2 z. \] (1)

As usual, this force has to be balanced by a pressure gradient, so in the steady state the vertical momentum equation becomes simply

\[ 0 = -\frac{\partial p}{\partial z} + \rho g_z. \] (2)

This can be integrated in certain cases, eg for the case of constant temperature or under the assumption of a polytropic atmosphere. However, the result is qualitatively always similar and only some numerical coefficients change.

A simple and instructive example is one where the density is constant (and equal to \( \rho_c \)) within the disc, \( |z| < H \), and zero outside, ie \( |z| > H \). In that case Equation (2) can be integrated as follows:

\[ -\int_0^H \frac{\partial p}{\partial z} dz = \int_0^H \rho \Omega^2 z dz. \] (3)

\[ -[p(H) - p(0)] = \frac{1}{2} \rho_c \Omega^2 H^2. \] (4)

Now, since \( p(H) = 0 \) and \( p(0) = \rho_c \), we have

\[ \frac{\rho_c}{\rho} = \frac{1}{2} \Omega^2 H^2. \] (5)

So, in this case there is a factor 1/2 on the right hand side.

Another simple example is the isothermal disc (isothermal in the vertical direction), where \( p = c_s^2 \rho_c \).

The solution for the vertical structure is

\[ \ln \rho = \ln \rho_c - \frac{\Omega^2 z^2}{2 c_s^2}. \] (6)

In this case the disc extends till infinity, so \( H \) has to be defined as the height where \( \rho \) has dropped below a certain value. If we take \( \rho = \rho_c \exp[-z^2/(2H^2)] \), then we have

\[ c_s^2 = \frac{\rho_c}{\rho} = \Omega^2 H^2, \] (7)

or

\[ c_s = \Omega H, \] (8)

which is the relation adopted here.

For the present purpose it will suffice to carry out the various vertical integrations by simply replacing

\[ \frac{\partial}{\partial z} \rightarrow \pm \frac{1}{H}, \quad z \rightarrow H, \] (9)

where \( H \) is some disc height and the sign depends on where the differentiated quantity increases or decreases with height. In the case of the pressure, which decreases with height, we have a minus sign, for example. Thus, we have \( p_c = \rho_c \rho_c H = \rho_c \Omega^2 H^2 \). Here the subscript \( c \) refers to central values, ie to the values at the midplane. Since the ratio \( p/\rho \) is the isothermal sound speed, \( c_s \), we have

\[ c_s = \Omega H. \] (10)

This is a relation that is strictly valid in the case of an isothermal atmosphere (see the box above). The
sound speed is related to the central temperature \( T_c \) via
\[
\frac{RT_c}{\mu} = c_s^2. \tag{11}
\]
These last two equations, (10) and (11), are the first two equations governing the structure of accretion discs. Next we need to find an expression that relates the temperature to some other variable. This can be done by considering the radiative equilibrium.

2.2 Radiative equilibrium

The disc temperature results from a balance between heating and cooling. The cooling comes from radiative losses. The decrease in thermal energy density is equal to the divergence of the radiative flux, which we introduced in connection with the Sun.

The source of heating in an accretion disc is less obvious. Ultimately, the energy released in the form of radiation comes from the potential energy that is liberated when matter falls towards the central object.

In discs heat is generated by friction. Think for example of a rapidly spinning motor saw. If you press too strongly to the wood it will become hot and start to burn. This is just because of frictional heat and thus proportional to the (dynamical) viscosity \( \mu \), which was introduced earlier in connection with friction experienced by a single particle. The frictional heat must also depend on the velocity gradient between adjacent rings of gas. However, the heat cannot depend on the sign of the shear, so it must be proportional to the square of the shear, so
\[
\text{frictional heat} = \text{dynamical viscosity} \times \text{(shear)}^2. \tag{12}
\]

In a rotating system the velocity gradient (ie \( \partial u / \partial x \)) has to be replaced by the angular velocity gradient multiplied by the cylindrical radius, ie shear = \( \varpi \partial \Omega / \partial \varpi \). Those things can be derived rigorously, but here we just try to motivate the various expressions qualitatively.

Thus, the heat generated per unit volume is
\[
\text{heat per unit volume} = \mu \left( \varpi \frac{\partial \Omega}{\partial \varpi} \right)^2. \tag{13}
\]
The dynamical viscosity is the product of gas density \( \varrho \) times the so-called kinematic viscosity \( \nu \), ie \( \mu = \varrho \nu \). As mentioned earlier, in accretion discs one often operates with vertically integrated quantities, so therefore we replace \( \varrho \) by \( \Sigma \). Furthermore, since a disc has two surfaces, the heat going into the upper disc plane is only one half of the total, so therefore we have for the
\[
\text{heat per unit area} = \frac{1}{2} \nu \Sigma \left( \frac{3}{2} \Omega \right)^2, \tag{14}
\]
where we have made use of the fact that thin discs rotate according to Kepler’s law, \( \Omega \sim \varpi^{-3/2} \).

This heat is lost by radiation and the rate of loss is equal to the divergence of the radiative heat flux,
\[
F_{\text{rad}}(z) = -\frac{16\sigma_{SB} T^4}{3\kappa \varrho} \frac{\partial T}{\partial z}, \tag{15}
\]
where \( \kappa \) is the opacity. The vertically integrated divergence of \( F_{\text{rad}} \) is given by \( F_{\text{rad}}(H) \) minus \( F_{\text{rad}}(0) \), but \( F_{\text{rad}}(0) \) vanishes because of symmetry reasons. Before replacing then all variables by vertically integrated quantities we write \( 4T^3 \partial T / \partial z = \partial T^4 / \partial z \), so therefore the
\[
\text{heat loss per unit area} = F_{\text{rad}}(H) = \frac{4\sigma_{SB} T^4}{3\kappa \Sigma}. \tag{16}
\]
So the radiative balance equation becomes
\[
\frac{4\sigma_{SB} T^4}{3\kappa \Sigma} = \frac{1}{2} \nu \Sigma \left( \frac{3}{2} \Omega \right)^2. \tag{17}
\]
Together with equations (10) and (11) this is the third equation governing the structure of accretion discs. We still need an equation that relates the accretion rate and the viscosity to the remaining variables. In fact, one of the equations not used yet is the equation of angular momentum conservation.

2.3 Angular momentum conservation

In general, the rate of change of the angular momentum (AM) density, \( \Sigma \Omega \varpi^2 \), is balanced by the negative divergence of the angular momentum flux, ie

\[
\frac{\partial}{\partial t} (\text{rate of change of AM density}) = -\nabla \cdot (\text{AM flux density}) \tag{18}
\]

The angular momentum flux has no net component in the vertical direction, only in the radial. The radial component of the angular momentum flux consists of two main contributions, one form the advection of angular momentum, ie

\[
\text{advected AM} = (2\pi \varpi) v_\varpi \left( \Sigma \varpi^2 \Omega \right) = -\dot{M} \varpi^2 \Omega. \tag{19}
\]

The mass flux \( \dot{M} \) through a ring at radius \( \varpi \) is equal to

\[
\dot{M} = -2\pi \int_{-\infty}^{\infty} \varpi \Sigma v_\varpi \, dz = -2\pi \varpi \Sigma v_\varpi, \tag{20}
\]

where the minus sign means that we count the mass flux positive if matter is accreted, ie when the radial velocity \( v_\varpi \) is negative.

The other contribution to the radial angular momentum flux is the viscous flux of angular momentum, ie

\[
\text{viscous AM flux} = -2\pi \varpi \left( \nu \Sigma \varpi^2 \frac{\partial \Omega}{\partial \varpi} \right). \tag{21}
\]

Note that the latter has a minus sign. This is because AM is transported down the gradient of angular velocity.

In the steady state the divergence of the sum of the two flux densities vanishes, so the sum of the two fluxes is constant,

\[
-\dot{M} \varpi^2 \Omega - 2\pi \varpi \left( \nu \Sigma \varpi^2 \frac{\partial \Omega}{\partial \varpi} \right) = C, \tag{22}
\]

ie equal to an integration constant \( C \) to be determined now.

Near the surface of the central object the angular velocity must match the angular velocity of the star, but because the viscosity is generally small, this will be very near the stellar radius \( R \). So, for \( \varpi \approx R \) we have

\[
-\dot{M} R^2 \Omega(R) = C. \tag{23}
\]

Plugging this into Equation (22) we have

\[
-\dot{M} \varpi^2 \Omega - 2\pi \varpi \nu \Sigma \varpi^2 \frac{\partial \Omega}{\partial \varpi} = -\dot{M} R^2 \Omega(R). \tag{24}
\]

Now we use Kepler’s law for \( \varpi \partial \Omega/\partial \varpi = -\frac{3}{2} \Omega \), so

\[
2\pi \nu \Sigma \varpi^2 \left( \frac{3}{2} \Omega \right) = \dot{M} \left[ \varpi^2 \Omega - R^2 \Omega(R) \right]. \tag{25}
\]

Using \( \Omega(R) R^2 = \sqrt{GM R} \) we have finally

\[
\nu \Sigma = \frac{\dot{M}}{3\pi} \left[ 1 - \left( \frac{R}{\varpi} \right)^{1/2} \right]. \tag{26}
\]
This is then the fourth equation needed to calculate the radial structure of accretion discs. Let us count the number of unknowns: $c_s$, $H$, $T_c$, $\Sigma$, $\kappa$ and $\nu$, so we have altogether 6 unknowns. We need two more equations for the opacity $\kappa$ and the viscosity $\nu$. Such equations are sometimes called “material equations”, because they describe the behaviour of the matter and are exchangeable, unlike three of the other equations with were based on physical conservation laws (vertical and radial momentum together with radiative equilibrium). Equation (11) was also a material equation, because it assumed a perfect gas. [If radiation pressure becomes dominant, for example, this equation will take the form $c_s^2 = (4\sigma_{SB}/c)(T_4^{2}H/\Sigma)$.

The expression for $\kappa$ is complicated, but in principle straightforward, because it is based on atomic physics which is well understood. In general, $\kappa$ is a function of $\rho$ and $T$ which has been tabulated. In many cases where only certain absorption processes are important simple formulae can be used. For example, if only free-free transitions are important we can use Kramer’s opacity

$$\kappa = \kappa_0 \rho T^{-7/2}, \quad (27)$$

where $\kappa_0 = 6.6 \times 10^{18} \text{ m}^5 \text{ K}^{7/2} \text{ kg}^{-2}$. [This value may well be up to 30 times larger if the gas is “metal rich”, i.e. a good electron supplier, so that bound-free processes become important as well.] This equation is valid at any point, but we shall apply it only to the midplane, so we adopt central values $\kappa = \kappa_0 \rho_0 T^{-7/2}$. For very hot gases electron scattering becomes the dominant absorption process. In that case $\kappa = 0.04 \text{ m}^2 \text{ kg}^{-1}$ is constant.

**Absorption processes in astrophysical plasmas.**

- **Electron scattering**: if an electromagnetic wave passes an electron the electric field makes the electron oscillate.
- **Free-free transitions**: if during its thermal motion a free electron passes an ion, the two charged particles form a system which can absorb and emit radiation.
- **Bound-free transitions**: a neutral hydrogen atom in its ground state is ionised by a photon.
- **Bound-bound transitions**: after absorption of a photon the electron jumps to a higher bound state, rather than leaving the atom altogether.
- **Negative Hydrogen ion**: a neutral hydrogen atom is polarised by a nearby charge and can then attract and bind another electron.

[Adapted from Kippenhahn and Weigert (1990).]

The expression for $\nu$ is more complicated, because the viscosity is due to some ill-understood turbulent processes. Molecular viscosity would be far too small to cope with the observed accretion rates, cf. Equation (26). By analogy with kinetic gas theory, where the viscosity equals the root-mean-square transport velocity (in general the sound speed) times a mean-free path, we assume that the turbulent viscosity also scales with the sound speed $c_s = \Omega H$ times a fraction $\alpha$ of the disc height $H$, which is the largest scale an eddy can have, so we write

$$\nu = \alpha \Omega H^2. \quad (28)$$

This expression is pretty crude, but it leads to a closed theory of accretion discs which agrees fairly well with observations. This expression was first introduced by Shakura & Sunyaev (1973) and $\alpha$ is therefore sometimes also called $\alpha_{SS}$. Also, don’t confuse this $\alpha$ with the $\alpha$ used in dynamo theory.
Summary of the equations governing the radial structure of accretion discs. There are six equations

\[ c_s = \Omega H, \]  
\[ \frac{RT_c}{\mu} = c_s^2 \left( = \frac{p_c}{\rho_c} = \frac{p_c}{\Sigma} H \right), \]  
\[ \frac{4 \sigma_{SB} T_c^4}{3 \kappa \Sigma} = \frac{1}{2} \nu \Sigma \left( \frac{3}{2} \Omega \right)^2, \]  
\[ \nu \Sigma = \frac{M}{3\pi} \left[ 1 - \left( \frac{R}{\varpi} \right)^{1/2} \right], \]  
\[ \kappa = \frac{\kappa_0}{H} T_c^{-7/2}, \]  
\[ \nu = \alpha \Omega H^2, \]

for the six unknowns \( c_s, H, T_c, \Sigma, \kappa \) and \( \nu \).

2.4 Solution of the disc equations

Solving the system of six algebraic equations in straightforward. First plug in (33) into (31) to eliminate \( \kappa \):

\[ \frac{4 \sigma_{SB} T_c^4}{3 \kappa \Sigma} \frac{15}{2} \frac{H}{\nabla} = \frac{1}{2} \nu \Sigma \left( \frac{3}{2} \Omega \right)^2. \]  

(35)

Eliminate \( T_c \) using (30) and move some numerical factors to the right-hand side.

\[ \left( \frac{R}{\mu} \right)^{15/2} (\Omega H)^{15} H = \frac{27 \kappa_0}{32 \sigma_{SB}} \nu \Sigma^3 \Omega^2. \]  

(36)

For abbreviation let us denote

\[ K \equiv \frac{27 \kappa_0}{32 \sigma_{SB}} \left( \frac{R}{\mu} \right)^{15/2} \]  

(37)

and eliminate \( \nu \) using (34) we have

\[ \Omega^{15} H^{16} = K \alpha \Sigma^3 \Omega^3 H^2, \]  

(38)

or

\[ H^{14} = K \alpha \Sigma^3 \Omega^{-12}. \]  

(39)

This is a relation between \( H \) and \( \Sigma \) alone, because the other variables are constant \( (K \alpha) \) or known functions of radius \( (\Omega) \). To solve then the whole system of equations we still need another relation between \( H \) and \( \Sigma \). This can be obtained by combining (32) and (34). For abbreviation let us denote

\[ \dot{m} = \frac{M}{3\pi} \left[ 1 - \left( \frac{R}{\varpi} \right)^{1/2} \right]. \]  

(40)

The term in squared brackets is usually called \( f^4 \), so

\[ f = \left[ 1 - \left( \frac{R}{\varpi} \right)^{1/2} \right]^{1/4}. \]  

(41)

Eliminating \( \nu \) from those two equations yields simply

\[ \alpha \Omega H^2 \Sigma = \dot{m}. \]  

(42)
Now eliminating $\Sigma$ from (39) and (42) yields
\[ H^{14} = \mathcal{K}\alpha \left( \dot{m}^3 \alpha^{-3} \Omega^{-3} H^{-6} \right) \Omega^{-12}, \quad (43) \]
or
\[ H^{20} = \mathcal{K}\alpha^{-2} \dot{m}^3 \Omega^{-15}, \quad (44) \]
or
\[ H = \mathcal{K}^{1/20} \alpha^{-1/10} \dot{m}^{3/20} \Omega^{-3/4}. \quad (45) \]

Before we proceed with calculating the remaining relations let us now look at some numerical values.

### 2.5 Numerical values

First, using $\sigma_{SB} = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$, $\kappa_0 = 6.6 \times 10^{18} \text{ m}^5 \text{ K}^{7/2} \text{ kg}^{-2}$, $\mathcal{R} = 8315 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$, and $\mu = 0.62$ we have
\[ \mathcal{K} = 8.8 \times 10^{56} \text{ m}^{20} \text{ s}^{-12} \text{ kg}^{-3}. \quad (46) \]

Note that $1 \text{ W} = 1 \text{ kg m}^2 \text{ s}^{-3}$, so $\sigma_{SB} = 5.67 \times 10^{-8} \text{ kg s}^{-3} \text{ K}^{-4}$. The exact value of $\alpha$ is unknown, because it depends on the nature of the turbulence, for which no theory exists at present. Numerical simulations however indicate that the lower limit is $\alpha = 0.01$, but it is conceivable that $\alpha$ may be much closer to unity. Fortunately, in many expressions $\alpha$ enters only with a relatively small power (see the box below).

Typical values for binaries are $\dot{M} = 10^{11} \text{ kg/s}$, so $\dot{m} \approx 10^{14} \text{ kg/s}$. We assume $\dot{M} = 1M_\odot = 2 \times 10^{30} \text{ kg}$ for the central mass. The keplerian angular velocity is then
\[ \Omega = \left( \frac{GM}{\varpi^3} \right)^{1/2} \approx \left( \frac{10^{-10} \times 10^{30}}{(10^8)^3} \right)^{1/2} \text{ s}^{-1} = 0.01 \text{ s}^{-1}. \quad (47) \]

The exact calculation (with $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ and $M = 2 \times 10^{30} \text{ kg}$) yields the value $\Omega = 0.012 \text{ s}^{-1}$. We can always calculate $\Omega$ for other values of the central mass $M$ and distance from the central mass $\varpi$, because we know that $\Omega$ scales with $M$ to the power $1/2$ and with $\varpi$ to the power $-3/2$, so we may write
\[ \Omega = 0.012 \left( \frac{M}{1M_\odot} \right)^{1/2} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-3/2} \text{ s}^{-1}. \quad (48) \]

Coming back to Equation (45), let us first calculate the value of the disc height $H$ for the reference values mentioned above. The rough calculation gives
\[ H = 10^{56/20} \times 10^{12 \times 3/20} \times 0.01^{-3/4} \text{ m} \approx 10^{2.9+1.8+1.5} \text{ m} = 10^{6.2} \text{ m}. \quad (49) \]

Again, a more precise calculation yields the value $H = 1.2 \times 10^9 \text{ m}$. To obtain the full dependence on $M$, $\dot{M}$ and $\varpi$ we simply restore the relevant powers from (45) and (48), so
\[ H = 1.2 \times 10^6 \times \alpha^{-1/10} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{3/20} \left( \frac{M}{1M_\odot} \right)^{-3/8} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{9/8} \text{ m}. \quad (50) \]

### 2.6 Temperature

Let us now calculate the scaling of the disc temperature. To do this we make use of relations (29) and (30)
\[ \frac{\mathcal{R} T_c}{\mu} = c_s^2 = \Omega^2 H^2. \quad (51) \]
Since we know $\Omega$ and $H$, we have

$$T = \frac{\mu}{R} \times 0.012^2 \left( \frac{M}{1 \text{M}_\odot} \right) \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-3} \times (1.2 \times 10^6)^2 \alpha^{-1/5} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{3/10} \left( \frac{M}{1 \text{M}_\odot} \right)^{-3/4} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{3/2} \text{ K},$$

and with $\mu/R = (0.62/8315) \text{ K s}^2 / \text{ m}^2 \approx 10^{-4} \text{ K s}^2 / \text{ m}^2$ we have

$$T = 1.5 \times 10^4 \times \alpha^{-1/5} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{3/10} \left( \frac{M}{1 \text{M}_\odot} \right)^{1/4} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-3/4} \text{ K}. \quad (53)$$

This is a temperature similar to that of the Sun. Indeed, in many respects the vertical structure of accretion discs is quite reminiscent of the vertical structure of stars. We can now use this relation to find the temperature at all radii. For example, if the central mass is a neutron star with radius $R = 10 \text{ km} = 10^{-1} \text{ m}$, we have a temperature near the surface of the star of $10^4 \times (10^4/10^6)^{-5/2} \text{ K} = 10^7 \text{ K}$.  

### 2.7 The surface density

To calculate $\Sigma$ we can use equations (32) and (34), so

$$\Sigma = \dot{m} \nu^{-1} = \dot{m} \alpha^{-1} \Omega^{-1} H^{-2}. \quad (54)$$

Using (48) and (50) we have

$$\Sigma = 1.1 \times 10^{12} \times \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right) \alpha^{-1} (0.012)^{-1} \left( \frac{M}{1 \text{M}_\odot} \right)^{-1/2} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{3/2} \times (1.2 \times 10^6)^{-2} \times \alpha^{-1/5} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{-3/10} \left( \frac{M}{1 \text{M}_\odot} \right)^{3/4} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-9/4} \text{ kg m}^{-2}. \quad (55)$$

This gives

$$\Sigma = 61 \times \alpha^{-4/5} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{7/10} \left( \frac{M}{1 \text{M}_\odot} \right)^{1/4} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-3/4} \text{ kg m}^{-2}. \quad (56)$$

### 2.8 The pressure

Using (29) and (30) we see that the pressure is given by

$$p_c = \Omega^2 H \Sigma \quad (57)$$

ie

$$p_c = (0.012)^2 \left( \frac{M}{1 \text{M}_\odot} \right) \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-3} \times 1.2 \times 10^6 \times \alpha^{-1/10} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{3/20} \left( \frac{M}{1 \text{M}_\odot} \right)^{-3/8} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{9/8} \times 61 \times \alpha^{-4/5} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{7/10} \left( \frac{M}{1 \text{M}_\odot} \right)^{1/4} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-3/4} \text{ kg m}^{-1} \text{ s}^{-2} \quad (58)$$

$$p_c = 1.1 \times 10^4 \times \alpha^{-9/10} \left( \frac{\dot{M} f^4}{10^{13} \text{ kg/s}} \right)^{-3/20} \left( \frac{M}{1 \text{M}_\odot} \right)^{7/8} \left( \frac{\varpi}{10^8 \text{ m}} \right)^{-21/8} \text{ kg m}^{-1} \text{ s}^{-2} \quad (59)$$
In the derivation we have omitted the contribution of the radiation pressure to the total pressure. In general we would have to replace the pressure by the sum of the gas pressure $p_{\text{gas}} = (R/\mu)T_0$ and radiation pressure, $p_{\text{rad}} = \frac{4\pi m}{3c} T^4$. If $p_{\text{rad}} \ll p_{\text{gas}}$ we can neglect the radiation pressure. To see whether this is the case we compute $p_{\text{rad}}$ for the standard disc solution:

$$p_{\text{rad}} = \frac{4\sigma SB}{3c} T^4 = 13 \times \alpha^{-4/5} \left( \frac{M f^4}{10^{13} \text{kg/s}} \right)^{6/5} \left( \frac{M}{1 M_\odot} \right) \left( \frac{\omega}{10^8 \text{m}} \right)^{-3} \text{kg m}^{-1} \text{s}^{-2}.$$  

(64)

This value is much smaller than $p_{\text{gas}}$ at the same radius. To see how this changes as we go further in let us calculate the ratio

$$\frac{p_{\text{rad}}}{p_{\text{gas}}} = 1.2 \times 10^{-3} \times \alpha^{1/10} \left( \frac{M f^4}{10^{13} \text{kg/s}} \right)^{7/20} \left( \frac{M}{1 M_\odot} \right)^{1/8} \left( \frac{\omega}{10^8 \text{m}} \right)^{-3/8}.$$  

(65)

Obviously, this ratio is small. Only for $\omega \approx 1.6 \text{ m}$ (ie way into the central object itself!) does it become comparable to unity.

2.9 The opacity

Similar to the pressure, the opacity too has (at least) two different contributions: Kramer’s opacity, $\kappa_{Kr}$, and the opacity for electron scattering, $\kappa_{es}$. The latter is constant ($= 0.04 \text{ m}^2 \text{kg}^{-1}$), ie independent of $\rho$ and $T$. In order to see when electron scattering becomes important, let us calculate $\kappa_{Kr}$ for the standard solution:

$$\kappa_{Kr} = \kappa_0 \Sigma H^{-1} T^{-7/2},$$  

(66)

$$\kappa_{Kr} = 0.82 \left( \frac{M f^4}{10^{13} \text{kg/s}} \right)^{-1/2} \left( \frac{M}{1 M_\odot} \right)^{-1/4} \left( \frac{\omega}{10^8 \text{m}} \right)^{3/4} \text{ m}^2 \text{kg}^{-1}.$$  

(67)

Note that this expression is independent of $\alpha$. We see that only for $\omega \leq 1.8 \times 10^6 \text{ m}$ does electron scattering become important.

3 Active Galaxies and Quasars

Much more should be said, but we there was not enough time to cover it all. Several points will just be mentioned very briefly. Most of the space is devoted to the nuclei of so-called active galaxies. They are the engine of quasars — quasi-stellar (point-like) objects, and are among the brightest objects in the sky.
Quasars have been somewhat of a mystery since they were discovered in 1963, but now we know that they are basically just accretion discs on a gigantic scale around a supermassive black hole. They are very far away, at cosmological distances, and have redshifts of $z = 0.1$ till $z = 5$. (Because the Universe is expanding, the further something is away, the faster it runs away from us, so the larger is the redshift. For $z = 0.16$, as in the quasar 3C273, the redshift is 16%, whilst for $z = 5$ the entire electromagnetic spectrum is rescaled by a factor of 5.) It is really because of those large distances that quasars look pointlike. However, now with the Hubble Space Telescope (HST) it has been possible to say that all quasars are really within more-or-less ordinary looking galaxies. Therefore we now believe that when the Universe was a few $10^8$ yr old, most galaxies must have gone through a phase where they had a quasar in their centre.

4 Active galactic nuclei

The luminosity of an AGN is about 100 times larger than the luminosity of ordinary galaxies. There are actually objects that are even brighter than AGNs, but only for a very short time. Those objects are gamma-ray bursters, whose nature we are only now beginning to understand, even though they have been detected over 20 years ago when people were looking for gamma ray bursts originating from H-bomb ignitions.

Typical masses of AGNs are around $10^8$ solar masses. It has long been a mystery how a luminosity of $10^{13}$ solar luminosities could be explained. The mass-luminosity relation for galaxies is approximately linear, ie $L \sim M^n$ with $n \approx 1$, so this would not suffice to explain the enormous luminosities of quasars. Stars have a nonlinear mass-luminosity relation, $L \sim M^3$ for main sequence stars, which could yield sufficient luminosity for objects of $10^8$ solar masses, if that relation was actually valid for quasars. However, there are serious stability problems when trying to explain so massive objects in terms of superluminous stars. In fact, for stars exceeding 50 solar masses the radiation pressure becomes so immense that it would blow away the star’s atmosphere if the luminosity exceeded the $L \sim M$ relation.

The reason why accretion discs can explain such high luminosities is because they can release huge amounts of binding energy. A body of mass $m$ that falls onto another object of mass $M$ gains kinetic energy that is equal to the potential energy, which is

$$E = \frac{GM}{R} m. \quad (68)$$

If mass falls in at a rate $\dot{M}$ the rate of energy release is

$$L \equiv \dot{E} = \frac{GM}{R} \dot{M}. \quad (69)$$

The smaller the central body, and the more massive it is, the larger will be the energy release. The smallest and yet massive bodies are black holes. The Schwarzschild radius is $2GM/c^2$, where $c$ is the speed of light, and if we identify this with $R$ we have

$$L \equiv \dot{E} = \frac{1}{2} c^2 \dot{M}. \quad (70)$$

For accretion rates of 2 solar masses per year this amounts to

$$L \equiv \dot{E} = 0.5 \, (3 \times 10^8)^2 \, \frac{2 \times 2 \times 10^{30}}{3 \times 10^7} \approx 10^{17} \, 10^{23} = 10^{40} \, \text{W},$$

which would fit the observed luminosities of quasars very well. ($10^{13}$ solar luminosities correspond to approximately $10^{13} \times 4 \times 10^{26} \, \text{W} = 4 \times 10^{39} \, \text{W}$.) The remarkable thing here is that such a mechanism can lead to energies very close to the rest mass energy, $Mc^2$. Note that nuclear fusion is much less efficient: here the energy generated is only $0.007 \times Mc^2$. In that sense one can say that black hole accretion discs are more powerful than nuclear fusion! In the following we consider the properties of such discs in more detail.
5 AGN discs

In this section we use the standard accretion disc equations, but rescaled to the parameters applicable to active galactic nuclei:

\[
\dot{M} = 10^{23} \text{ kg s}^{-1}, \quad M = 10^8 M_\odot = 2 \times 10^{38} \text{ kg}, \quad c = 10^{12} \text{ m} \quad (\text{for AGNs}).
\] (72)

An important length scale here is the Schwarzschild radius, \( R_S = GM/(2c^2) \), where \( c = 3 \times 10^8 \text{ m/s} \) is the speed of light. It would not be meaningful to consider disc radii smaller than \( R_S \). For \( 10^8 \text{ solar masses} \), this radius is \( R_S = 7.4 \times 10^{10} \text{ m} \).

Let us first calculate the orbital frequency for those values:

\[
\Omega = \sqrt{\frac{GM}{c^3}} = \sqrt{\frac{6.67 \times 10^{-11} \times 2 \times 10^{38}}{(10^{12})^3}} = 1.2 \times 10^{-4} \left( \frac{M}{10^8 M_\odot} \right)^{1/2} \left( \frac{c}{10^{12} \text{ m}} \right)^{-3/2} \text{ s}^{-1}.
\] (73)

In order to obtain the disc height \( H \) for the parameters \( \Omega \) we just plug in those numbers into Equation (50)

\[
H_{\text{AGN}} = 1.2 \times 10^6 \times \left( \frac{10^{23} \text{ kg/s}}{10^{13} \text{ kg/s}} \right)^{3/20} \left( \frac{10^8 M_\odot}{1 M_\odot} \right)^{-3/8} \left( \frac{10^{12} \text{ m}}{10^8 \text{ m}} \right)^{9/8} \text{ m} = 1.2 \times 10^6 \text{ m}.
\] (74)

Restoring now the full dependence on the various parameters we have

\[
H = 1.2 \times 10^9 \times \alpha^{-1/10} \left( \frac{\dot{M}}{10^{23} \text{ kg/s}} \right)^{3/20} \left( \frac{M}{10^8 M_\odot} \right)^{-3/8} \left( \frac{c}{10^{12} \text{ m}} \right)^{9/8} \text{ m}.
\] (75)

For the temperature we calculate first the value for the AGN parameters using Equation (61)

\[
T_{\text{AGN}} = 1.5 \times 10^4 \times \left( \frac{10^{23} \text{ kg/s}}{10^{13} \text{ kg/s}} \right)^{3/10} \left( \frac{10^8 M_\odot}{1 M_\odot} \right)^{1/4} \left( \frac{10^{12} \text{ m}}{10^8 \text{ m}} \right)^{-3/4} \text{ K} = 10^6 \text{ K}.
\] (76)

Again, restoring the full dependence on parameters we have

\[
T = 1.5 \times 10^6 \times \alpha^{-1/5} \left( \frac{\dot{M}}{10^{23} \text{ kg/s}} \right)^{3/10} \left( \frac{M}{10^8 M_\odot} \right)^{1/4} \left( \frac{c}{10^{12} \text{ m}} \right)^{-3/4} \text{ K}.
\] (77)

Now for the surface density, we have from Equation (62)

\[
\Sigma_{\text{AGN}} = 61 \times \left( \frac{10^{23} \text{ kg/s}}{10^{13} \text{ kg/s}} \right)^{7/10} \left( \frac{10^8 M_\odot}{1 M_\odot} \right)^{1/4} \left( \frac{10^{12} \text{ m}}{10^8 \text{ m}} \right)^{-3/4} \text{ kg m}^{-2} = 6.1 \times 10^7 \text{ kg m}^{-2}.
\] (78)

and so the general formula is

\[
\Sigma = 6.1 \times 10^7 \times \alpha^{-4/5} \left( \frac{\dot{M}}{10^{23} \text{ kg/s}} \right)^{7/10} \left( \frac{M}{10^8 M_\odot} \right)^{1/4} \left( \frac{c}{10^{12} \text{ m}} \right)^{-3/4} \text{ kg m}^{-2}.
\] (79)

Finally for the pressure we have from Equation (63)

\[
p_{\text{c AGN}} = 1.1 \times 10^4 \times \left( \frac{10^{23} \text{ kg/s}}{10^{13} \text{ kg/s}} \right)^{-3/20} \left( \frac{10^8 M_\odot}{1 M_\odot} \right)^{7/8} \left( \frac{10^{12} \text{ m}}{10^8 \text{ m}} \right)^{-21/8} \text{ kg m}^{-1} \text{s}^{-2} = 1.1 \times 10^9 \text{ kg m}^{-1} \text{s}^{-2}
\] (80)

\[
p_{\text{c}} = 1.1 \times 10^9 \times \alpha^{-9/10} \left( \frac{\dot{M}}{10^{23} \text{ kg/s}} \right)^{-3/20} \left( \frac{M}{10^8 M_\odot} \right)^{7/8} \left( \frac{c}{10^{12} \text{ m}} \right)^{-21/8} \text{ kg m}^{-1} \text{s}^{-2}
\] (81)
In this case the ratio of radiation pressure and gas pressure is

\[ \frac{p_{\text{rad}}}{p_{\text{gas}}} = 1.2 \times \alpha^{1/10} \left( \frac{\dot{M} f^4}{10^{23} \text{kg/s}} \right)^{7/20} \left( \frac{M}{10^8 M_\odot} \right)^{1/8} \left( \frac{\varpi}{10^{12} \text{m}} \right)^{-3/8} \]  

(82)

Note that this ratio is no longer small. Clearly, effects of the radiative pressure begin to be important and should be taken into account for more accurate considerations.

Electron scattering becomes important when the Kramer opacity drops below the value 0.04 m\(^2\) kg\(^{-1}\). Rescaling Equation (67) for values relevant for AGNs we have

\[ \kappa_{\text{Kr}} = 8.2 \times 10^{-5} \times \left( \frac{\dot{M} f^4}{10^{23} \text{kg/s}} \right)^{-1/2} \left( \frac{M}{10^8 M_\odot} \right)^{-1/4} \left( \frac{\varpi}{10^{12} \text{m}} \right)^{3/4} \text{ m}^2 \text{ kg}^{-1}. \]  

(83)

Thus, electron scattering is always important. Only for \( \varpi \geq 4 \times 10^{15} \text{ m} \) does Kramer’s opacity dominate over electron scattering. Since electron scattering is important in most parts of the disc we give for this case the solution in the box below.

Accretion disc solution for the case when electron scattering dominates over Kramer’s opacity:

\[ H = 2.4 \times 10^9 \times \alpha^{-1/10} \left( \frac{\dot{M} f^4}{10^{23} \text{kg/s}} \right)^{1/5} \left( \frac{M}{10^8 M_\odot} \right)^{-7/20} \left( \frac{\varpi}{10^8 \text{m}} \right)^{21/20} \text{ m}. \]  

(84)

\[ T = 6.2 \times 10^6 \times \alpha^{-1/5} \left( \frac{\dot{M} f^4}{10^{23} \text{kg/s}} \right)^{2/5} \left( \frac{M}{10^8 M_\odot} \right)^{3/10} \left( \frac{\varpi}{10^8 \text{m}} \right)^{-9/10} \text{ K}. \]  

(85)

\[ \Sigma = 1.5 \times 10^7 \times \alpha^{-4/5} \left( \frac{\dot{M} f^4}{10^{23} \text{kg/s}} \right)^{3/5} \left( \frac{M}{10^8 M_\odot} \right)^{1/5} \left( \frac{\varpi}{10^8 \text{m}} \right)^{-3/5} \text{ kg m}^{-2}. \]  

(86)

\[ \kappa_{\text{Kr}} = 7 \times 10^{-8} \times \left( \frac{\dot{M} f^4}{10^{23} \text{kg/s}} \right)^{-1} \left( \frac{M}{10^8 M_\odot} \right)^{-1/2} \left( \frac{\varpi}{10^{12} \text{m}} \right)^{3/2} \text{ m}^2 \text{ kg}^{-1}. \]  

(87)

With this expression, only for \( \varpi \geq 7 \times 10^{15} \text{ m} \) does Kramer’s opacity dominate over electron scattering.