Lecture 3: Stellar Winds and Blast Waves

Astrophysical flows are usually highly compressible. There are some specific effects resulting from that, one of which concerns the formation of winds and other types of outflows. The general principle of how a flow can attain supersonic speeds in a smooth manner (without going through a shock) can best be explained with the example of the Laval nozzle that is used in all rocket motors where the exhaust velocity can be 2000...3000 m/s, which is up to ten times the speed of sound.

We follow here the book by Shore, Chapter 8, p 257. We summarize what’s written there, and add some details that help to understand the problem. We begin this section by first discussing the general issue of how a flow can be made supersonic. We then apply this to winds and outflows from stars and discs and mention other circumstances where this formalism can be used. A rather important application is jets, i.e. outflows from accretion discs that become highly collimated. The collimation is probably due to the magnetic field, but this is very much an open research topic at the moment. The highly relativistic version of the wind problem is relevant for understanding outflows from active galactic nuclei (AGNs) and the gamma-ray burst (GRB) phenomenon.

1 The analogy with the Laval nozzle

We are used to think that the flow through a pipe is becoming faster when the cross-section becomes smaller (garden hose experiment!). However, this is only true when the fluid is incompressible. That’s often not a good approximation in astrophysics. The following problem illustrates this.

In a steady state the mass flux through a pipe is conserved, i.e.

$$\rho v_x S = \text{const.}$$

(1)

Here, $S = S(x)$ is the cross-section, $v_x(x)$ the streamwise velocity and $\rho(x)$ the density. This equation follows from integrating the equation $\nabla \cdot (\rho \mathbf{v}) = 0$ over the surface of a pipe that becomes narrower, and by applying Gauss’ divergence theorem; see Figure 1.

Indeed, if $\rho$ is constant, (1) shows that a larger cross-sectional area $S$ implies a smaller speed $v_x$. But what if $\rho$ actually decreases? In the following we will see an example where supersonic speeds can be achieved by decreasing density and increasing the cross-sectional area in a suitable manner.

The velocity is obtained by solving the steady, one-dimensional, isothermal momentum equation without any extra forces,

$$\frac{d}{dx} \left( \frac{1}{2} v_x^2 + \frac{1}{2} \rho S \right) = \delta v_x \frac{d v_x}{dx} = -c_s^2 \frac{d \ln \rho}{dx},$$

(2)

We now use Equation (1), differentiate logarithmically, i.e.

$$0 = \frac{1}{\rho v_x S} \frac{d}{dx} (\rho v_x S) = \frac{d}{dx} \ln (\rho v_x S) = \frac{d \ln \rho}{dx} + \frac{d \ln |v_x|}{dx} + \frac{d \ln S}{dx},$$

(3)

so we have

$$\frac{d \ln \rho}{dx} = - \frac{d \ln |v_x|}{dx} - \frac{d \ln S}{dx}.$$  

(4)

Using that in Equation (2) yields

$$v_x \frac{d v_x}{dx} = -c_s^2 \left( - \frac{d \ln |v_x|}{dx} - \frac{d \ln S}{dx} \right).$$

(5)

We move the first term on the rhs to the left and on the left we write

$$v_x^2 \frac{d v_x}{dx} = c_s^2 \frac{d \ln |v_x|}{dx}.$$  

(6)

This yields the equation

$$\left( v_x^2 - v_0^2 \right) \frac{d \ln |v_x|}{dx} = c_s^2 \frac{d \ln S}{dx},$$

(7)
Fundamental discoveries about Space

The Parker Solar Probe is currently on its way to the sun. It was launched by NASA in 2018 and its first results were reported just before Christmas. This probe is the first to be named after a living person – Eugene N. Parker, professor emeritus at the University of Chicago, USA.

Eugene Parker is responsible for several fundamental discoveries about the gases that surround the sun and other stars. He has also developed the theory of how the solar wind arises and how magnetic fields arise and change in space. When he initially presented his theories, over fifty years ago, they were strongly challenged, but were later confirmed through observations from spacecrafts.

Eugene Parker was the first person to realise that the sun is not in equilibrium, as was previously thought. Quite the opposite, it releases mass; the charged gas of ions and electrons that makes up the sun’s “atmosphere” is expanding as a solar wind that stretches throughout our planetary system. His ideas are also the foundation for all the forecasts about the space weather, which can disrupt satellites and cause power outages here on Earth.

He will now receive the Crafoord Prize in Astronomy “for pioneering and fundamental studies of the solar wind and magnetic fields from stellar to galactic scales”.

Eugene Parker was stunned into silence when he was told about the award:

“It took my breath away. I didn’t do anything for a few minutes. I of course knew about the Crafoord Prize, so I was surprised, pleasantly so.”
where $S(x)$ is the cross-sectional area of the nozzle, which is a known function of $x$. To obtain $v_x(x)$ we can integrate
\[
\frac{d \ln |v_x|}{dx} = c_s^2 \frac{d \ln S}{dx} \frac{1}{v_x^2 - c_s^2}.
\]
but in order for the solution to be regular when $|v_x| = c_s$ we have to require that the nominator vanishes at the same point. Thus, the critical point is where $S(x)$ has a minimum, because then $d \ln S/dx = 0$, which must be where $v_x = c_s$. Define $\rho v_x S = \dot{M}$ and integrate to obtain an implicit equation for $v(x)$:
\[
\frac{1}{2} c_s^2 \ln |v_x| - c_s^2 \ln S = E - c_s^2 \ln \dot{M} \equiv E'.
\]
This is just the usual Bernoulli equation that we have encountered elsewhere. We can now determine $E'_{\text{crit}}$ by applying the known values of $r$ and $v_r$ at the critical. For $c_s = \dot{M} = 1$ we find $E'_{\text{crit}} = \frac{1}{2} c_s^2 - c_s^2 \ln S_{\text{min}}$.

2 The isothermal wind problem

We now consider the steady, isothermal wind problem. We adopt spherical polar coordinates, $(r, \theta, \phi)$, but assume spherical symmetry in this case, so $\partial / \partial \theta = \partial / \partial \phi = 0$. The continuity and Euler equations are then
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho v_r \right) = 0, \quad (10)
\]
\[
v_r \frac{dv_r}{dr} = c_s^2 \frac{d \ln \rho}{dr} - \frac{GM}{r^2}. \quad (11)
\]
These equations can be brought into the form
\[
(v_r^2 - c_s^2) \frac{d \ln |v_r|}{dr} = \frac{2 c_s^2}{r} - \frac{GM}{r^2}. \quad (12)
\]
There is a critical point $r_*$ where the flow becomes transonic. At that point the left hand side and the right hand side of the equation must vanish simultaneously, i.e.
\[
\frac{2 c_s^2}{r} = \frac{GM}{r^2}. \quad (13)
\]
This yields for the value of the critical point
\[
r_* = \frac{GM}{2 c_s^2} \quad (14)
\]
Assuming $c_s = 100 \text{ km/s}$, and using $G \approx 7 \times 10^{-11} \text{ m}^3 \text{s}^{-2} \text{kg}^{-1}$, $M = 2 \times 10^{30} \text{ kg}$, $1 \text{ AU} = 1.5 \times 10^{11} \text{ m}$, we have $r_* \approx 0.05 \text{ AU}$.

Figure 1: In the steady case we have $\nabla \cdot \rho \mathbf{v} = 0$. Using Gauss’ divergence theorem it follows that $\int \rho \mathbf{v} \cdot d \mathbf{S} = 0$. The two surfaces, $S_1$ and $S_2$, are the only places where the $\mathbf{v} \cdot d \mathbf{S} \neq 0$. 

\[
\frac{1}{2} c_s^2 \ln |v_x| - c_s^2 \ln S = E - c_s^2 \ln \dot{M} \equiv E'.
\]
Figure 2: Cross-sectional area of a laval nozzle. If the flow is able to reach the sound speed at the point of minimal cross-section, it must go supersonic behind that point.

The steady wind problem is characterized by two integrals of motions:

\[ \frac{1}{2} v_r^2 + c_s^2 \ln \rho - \frac{GM}{r} = \mathcal{E}, \]  
\[ \dot{M} = 4 \pi r^2 \rho v_r, \]  
see Eq. (9) in Shore’s book. From this we get

\[ \frac{1}{2} v_r^2 - c_s^2 \ln v_r - 2 c_s^2 \ln r - \frac{GM}{r^2} = \mathcal{E}' \]  
\[ \dot{M} = 4 \pi r^2 \rho v_r, \]  
where \( \mathcal{E}' = \mathcal{E} - c_s^2 \ln(\dot{M}/4\pi) \). (Note that the \( C \) in Shore’s book changes its meaning all the time).

To find the energy for the critical solution we assume a critical radius \( r_\ast \), so \( c_s = \sqrt{GM/2r_\ast} \). At \( r = r_\ast \) we have \( v = c_s \). This yields \( \mathcal{E} \) for the critical solution that goes through the critical point:

\[ \mathcal{E}_{\text{crit}}' = \frac{1}{2} c_s^2 - c_s^2 \ln c_s - 2 c_s^2 \ln r_\ast - \frac{GM}{r_\ast} \]  
(18)

We may find the solution through iteration:

\[ v_{\text{new}} = \sqrt{2(\mathcal{E}_r + c_s^2 \ln v)} \]  
(19)

where \( \mathcal{E}_r \equiv \mathcal{E}' + 2c_s^2 \ln r_\ast + GM/r_\ast \) is introduced for convenience. As usual, we can get another solution from the inverse iteration formula

\[ v_{\text{new}} = \exp \left( \frac{1}{2} v^2 - \mathcal{E}_r \right) \]  
(20)

For \( \mathcal{E}' < \mathcal{E}_{\text{crit}}' \) the upper solutions are obtained from Equation (19) whilst the lower solutions are obtained by iterating Equation (20). The critical solution, as well as two more solutions are shown in Figure 3.

Another possibility of obtaining a graphical representation of the possible wind solutions is to do a contour plot of \( \mathcal{E} \) as a function of \( r \) and \( v_r \); see Figure 4.

Note that the iteration process always yields that solution that has a minimum or a maximum at \( r = r_\ast \). To get the physically sensible solution one needs to reconnect the right branches by hand, This is done in Figure 5, where we have plotted the critical solution together with the density profile.

3 The time-dependent wind problem

It is illuminating to consider the time-dependent problem. For example, one may wonder what happens when one were to increase the density or the velocity at the bottom. The governing equations are

\[ \frac{\partial \ln \rho}{\partial t} = -v_r \frac{\partial \ln \rho}{\partial r} - 2 \frac{v_r}{r} - \frac{\partial v_r}{\partial r} \]  
(21)

\[ \frac{\partial v_r}{\partial t} = -v_r \frac{\partial v_r}{\partial r} - c_s \frac{\partial \ln \rho}{\partial r} - \frac{GM}{r^2} \]  
(22)
Figure 3: The critical wind solution, together with two other solutions for different values of $E'$. Here we use $r_*=1$, $GM=1$, and so $c^2_s=0.5$.

Since the flow comes from the inner boundary at $r=r_0$ we have to specify boundary conditions at $r=r_0$. Video animations of numerical experiments show that specifying the value of $v_r$ at the lower boundary is inconsistent. The solution has a strong desire to come back to the steady solution discussed above. On the other hand, the density on the lower boundary may well be specified arbitrarily. The solution has then just another value of $\dot{M}$.

The solution can actually be reversed and then we have the problem of spherically symmetric accretion (Bondi accretion). This is the reason why one should always write $\ln |v_r|$.

The trick is to write the equations in the form $(v_r^2-c_s^2)d\ln |v_r|/dx = \text{something}$. This something then tells us where the critical point is. Once we know that, we plug that back into the integrated momentum equations to obtain an implicit equation for $v_r$ that goes through the critical point.

4 Plotting wind solutions

The phenomenon of a stellar wind is just as counterintuitive as that of a siphon flow: how can a flow go first uphill before falling down on the other side? This is really because of the gravitational pull it receives once it has been pushed over the top. The astrophysical analogy is that of a gas flow in a binary from one star to another over the peak of their coming potential. Again, a steady flow develops once it becomes transonic at the peak of the potential.

Another analogy is that of the de Laval nozzle.\footnote{Gustaf de Laval (1845–1913) was a Swedish engineer and inventor. Since 1886, he was a member of the Royal Academy of Sciences and was an elected member of parliament from 1888 to 1890.} If one is able to push the gas through the nozzle fast enough so that it reaches supersonic speeds and one lets it expand afterwards, mass continuity requires
The contour levels are equidistant in steps of 0.2, symmetrically about the critical value $\mathcal{E}' = -0.576713$. Again, $r_\ast = 1$, $GM = 1$, and so $c_s^2 = 0.5$.

The velocity to continue accelerating while the density drops even further. What is remarkable here is that one crosses the “sound barrier” without producing a shock.

It sounds still strange, so how can we see that it really works? The best is to solve the equations,
which can be done using the Bernoulli equation\(^2\) and the constancy of mass flux. The latter is given by

\[ S \rho u = \text{const}, \]  

(23)

where \(S\) is the surface area, \(\rho\) is the density, and \(u\) is the velocity. The Bernoulli equation reads

\[ \frac{1}{2} u^2 + P + \phi = \text{const}, \]  

(24)

where \(P\) is the (modified) pressure, and \(\phi\) is the gravity potential. Here we see that \(P\) decreases when \(u\) increases. We also see that the sum \(\frac{1}{2} u^2 + P\) increases when we drop deeper into the potential well.

To simplify matters, we assume that the temperature is everywhere is the same (the gas is isothermal). Without going into details, we just state that in that case we have

\[ P = c^2 \ln \rho, \]  

(25)

where \(c\) is the sound speed. Using this, we can now eliminate \(\ln \rho\) from Equations (23) and (24). Here we make use of a trick by taking the logarithm of Equation (23), which gives

\[ \ln S + \ln \rho + \ln u = \text{const}, \]  

(26)

and then obtain

\[ \frac{1}{2} u^2 - c^2 (\ln S + \ln u) + \phi = \text{const}. \]  

(27)

The values of the constants in the various equations are not the same, but that does not matter.

Let us now consider some examples. In the case of the isothermal solar wind, the relevant surface area is the surface of a sphere, \(4 \pi r^2\), where \(r\) is the radius. The gravitational potential is \(\phi = -GM/r\), where \(G\) is Newton’s constant and \(M\) is the mass of the Sun. Setting the speed of sound to unity, and assuming also the product \(GM\) to be unity, we have

\[ C(r, u) = \frac{1}{2} u^2 - \ln r^2 - \ln u - \frac{1}{r}. \]  

(28)

To find solutions, we can just plot contours of \(C(r, u)\) as a function of \(r\) and \(u\); see Figure 6. In this way, we have produced a graphical solution to an otherwise rather complicated equation!

Interestingly, the solution works also for negative \(u\). Therefore, we should really replace \(\ln u\) by \(\ln |u|\). Negative flows correspond to inflows and were first derived by Herman Bondi (1952, MNRAS 112, 195). All these similarities and analogies were discovered much later after Parker’s work of 1958.

Next, we consider a flow over a quadratic potential of the form \(\phi = \text{const} - x^2\), so we consider here a flow in an \(xz\) plane. The result is independent of \(z\) and depends just on \(x\) and so we plot contours of

\[ C(x, u) = \frac{1}{2} u^2 - \ln u - x^2 \]  

(29)

in the \((x, u)\) plane; see Figure 7. In the case of a nozzle with the shape proportional to \(x^2/(1 + x^2)\), we have

\[ C(x, u) = \frac{1}{2} u^2 - \ln u - \frac{x^2}{1 + x^2}. \]  

(30)

This result is shown in Figure 8.

## 5 Roche-lobe overflow

There are many semi-detached binary stars where mass flows from the secondary to the primary. Somewhere between the two stars there is the \(L_1\) point, which is an equilibrium point where the forces from both stars and the centrifugal force all balance (see the box). Near that point the gravitational potential \(\Phi\) can be approximated by

\[ \Phi = \Phi_0 - \frac{1}{2} \Phi_2 x^2, \]  

(31)

\(^2\)In his book of 1752, the Swiss mathematician and physicist Daniel Bernoulli (1700-1782) deduced that the pressure decreases when the flow speed increases. [The famous Bernoulli distribution, however, is due to his nephew Jacob Bernoulli (1655-1705).]

\(^3\) In die Bernoulli equation, the modified pressure is given by \(P = \int dp/\rho\). Assuming a gas with an isothermal equation of state, the pressure is \(p = \rho c^2\), and since \(dp/\rho = d \ln \rho\), we have \(P = c^2 \ln \rho\).
where $x$ is the coordinate in the direction from the secondary to the primary; see Figure ?? A derivation is given in the Appendix.

Assume that the Coriolis force can be neglected, and that the flow across the point $x = 0$ can be described by the stationary, the isothermal, one-dimensional fluid equations take the form

$$\frac{d}{dx}(\rho v_x) = 0,$$

(32)

$$v_x \frac{dv_x}{dx} = -c_s^2 \frac{d \ln \rho}{dx} - \frac{d \Phi}{dx},$$

(33)

where $c_s = \text{const}$ is the speed of sound.

We rewrite the continuity equation in the form

$$\frac{d \ln \rho}{dx} = -\frac{d \ln |v_x|}{dx},$$

(34)

plug this into the momentum equation

$$v_x \frac{dv_x}{dx} = -c_s^2 \frac{d \ln \rho}{dx} - \frac{d \Phi}{dx},$$

(35)

and use $v_x \frac{dv_x}{dx} = v_x^2 \frac{d \ln |v_x|}{dx}$ to obtain

$$(v_x^2 - c_s^2) \frac{d \ln |v_x|}{dx} = \frac{d \Phi}{dx},$$

(36)

or

$$(v_x^2 - c_s^2) \frac{d \ln |v_x|}{dx} \approx -2\Phi_2(x - x_{L1}),$$

(37)
Figure 7: Contours of $C(x, u)$.

Figure 8: Contours of $C(x, u)$ for a nozzle.

This shows that the flow becomes supersonic at the $L_1$ point. This is therefore automatically also the
critical point. To find an implicit algebraic equation for \( v_x \) we integrate once and use \( \rho v_x = \dot{M} \)

\[
\frac{1}{2} v_x^2 - c_s^2 \ln |v_x| - \frac{1}{2} \Phi_2 x^2 = \mathcal{E} - c_s^2 \ln \dot{M} - \Phi_0 \equiv \mathcal{E}',
\]

(38)

where \( \mathcal{E} \) and \( \mathcal{E}' \) are integration constants. At the critical point we have

\[
\mathcal{E}'_{\text{crit}} = \frac{1}{2} c_s^2 - c_s^2 \ln c_s.
\]

(39)

We don’t know the values of \( c_s \) and \( \dot{M} \). For the purpose of the following discussion let’s assume \( c_s = \dot{M} = 1 \). This gives \( \mathcal{E}'_{\text{crit}} = 1/2 - 2.5 = -2 \). In the following plot we give the graph of the critical solution and compare with solutions for \( \mathcal{E}' = -1 \) (dotted line) and \( \mathcal{E}' = -3 \) (dashed line). For \( \mathcal{E}' < \mathcal{E}'_{\text{crit}} \) there are no solutions at the \( L_1 \) point, so this will not be a physical solution. For \( \mathcal{E}' > \mathcal{E}'_{\text{crit}} \) the solution never goes through the sonic point either and stays all the time supersonic or subsonic.

We now proceed and rewrite Equation (38) by substituting the integration constant using Equation (39), i.e.

\[
\frac{1}{2} v_x^2 - c_s^2 \ln |v_x| - \frac{1}{2} \Phi_2 x^2 = \frac{1}{2} c_s^2 - c_s^2 \ln c_s.
\]

(40)

or

\[
\frac{1}{2} (v_x^2 - c_s^2) - c_s |v_x| + \frac{1}{2} \Phi_2 x^2 = 0.
\]

(41)

This implicit equation can be solved using the approximation

\[
\ln(|v_x|/c_s) \approx (|v_x|/c_s) - 1
\]

(42)

near the critical point, \((|v_x|/c_s) \approx 1\). We thus obtain a quadratic equation,

\[
\frac{1}{2} (v_x^2 - c_s^2) - c_s |v_x| + \frac{1}{2} \Phi_2 x^2 = 0,
\]

(43)

or

\[
v_x^2 - 2c_s |v_x| + c_s^2 - \Phi_2 x^2 = 0.
\]

(44)

There are two solutions to this quadratic equation,

\[
|v_x|_{1,2} = c_s \pm \sqrt{c_s^2 - \Phi_2 x^2},
\]

(45)

or

\[
|v_x|_{1,2} = c_s \pm \sqrt{\Phi_2 x}.
\]

(46)

Near the critical point the pressure gradient is given by

\[
c_s^2 \frac{d\rho}{dx} = -\rho \left( v_x \frac{dv_x}{dx} + \frac{d\Phi}{dx} \right),
\]

(47)

or

\[
c_s^2 \frac{d\rho}{dx} = -\rho \left( c_s \sqrt{\Phi_2 - \Phi_2} \right),
\]

(48)

or, because \( |x| \ll 1 \),

\[
c_s^2 \frac{d\rho}{dx} \approx -\rho c_s \sqrt{\Phi_2}.
\]

(49)

Assuming that \( \Phi_2 = 25 \text{ s}^{-2} \) and \( c_s = 10 \text{ km/s} \), the value of the acceleration, \(-c_s^2 d\ln \rho/dx\), at the critical point is around \( 50 \text{ km/s}^{-2} = 5 \times 10^4 \text{ m/s}^{-2} \). The acceleration \( GM/R^2 \) of a single star with \( M = 2 \times 10^{30} \text{ kg} \) at distance \( R = 10^7 \text{ m} \) is around

\[
\frac{GM}{R^2} = \frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \times 2 \times 10^{30} \text{ kg}}{10^{14} \text{ m}^2} = 1.4 \times 10^6 \text{ m/s}^{-2},
\]

(50)

which is about 30 times larger than acceleration from the pressure gradient at the critical point.
6 What drives the wind? The need for a nonstatic corona

The following consideration illustrates that there cannot be a static solar corona. In the solar corona heat is transported via conduction (as opposed to radiation or convection). In the corona (Spitzer-type) heat conduction is important, so the conductivity $K$ satisfies

$$K = K_0 \left( \frac{T}{T_0} \right)^{5/2}. \quad (51)$$

The heat flux is then given by

$$F = -K \nabla T. \quad (52)$$

In order to have a steady state (vanishing divergence of the heat flux, i.e. $\nabla \cdot F = 0$) we have to require that the temperature satisfies

$$T = T_0 \left( \frac{r}{r_0} \right)^{-2/7}, \quad (53)$$

where $T_0$ is the temperature at the base of the corona at $r = r_0$. This is significantly shallower than the temperature for a polytropic stratification, which would imply a $r^{-1}$ behavior with

$$T_{\text{poly}} = \left( 1 - \frac{1}{\gamma} \right) \frac{\mu GM}{R} \frac{r}{r_0}. \quad (54)$$

Assuming a perfect gas, i.e. $p = R T \rho$ one can show that

$$p = p_0 \exp \left\{ \frac{7r_0}{5H_0} \left[ \left( \frac{r}{r_0} \right)^{-5/7} - 1 \right] \right\}. \quad (55)$$

Here $p_0$ is the pressure at the base of the corona and $H_0$ is the scale height of the pressure, i.e. the distance over which the pressure changes by a factor of $e$. The problem with this expression is that at infinity, $r \to \infty$, the pressure remains finite,

$$p(\infty) = p_0 \exp \left\{ \frac{7r_0}{5H_0} \right\}. \quad (56)$$

In fact, for the sun $H_0$ is comparable with the solar radius, $r_0$, and so $p(\infty)$ is just somewhat smaller than $p_0$, which is definitely quite unrealistic. For a correct solution the pressure must become comparable with the interstellar pressure, which is orders of magnitude smaller than the pressure near the sun. This inconsistency leads then to the idea of a nonstatic corona with an outflow.