Lecture 1: Fluids in astrophysical environments, hydrodynamic equations, specific entropy, hydrostatic equilibrium

1 Fluids in astrophysical environments

Astrophysics is concerned with phenomena in the sky that can be studied using mathematics and the law of physics. Many observed phenomena in the Universe are related to fluid dynamics and electromagnetism (stellar variability and rotation, solar sunspot cycle, stellar and galactic jets, galactic radio emission, cosmological expansion, etc). In this course, topics in modern astrophysical fluid dynamics will be discussed in order to obtain a conceptual understanding of the various processes involved. Their mathematical description will be introduced at an elementary level. The governing equations will be derived within the context they occur in.

2 Fluid dynamics

Fluid dynamics is the collective description of the flow of a large number of particles. Familiar examples of fluid dynamics include for example the flow of water in a river, surface waves in the sea, and of course the wind in the air. More complicated flow patterns on Earth are generated by obstructions due to solid bodies (buildings in a city, trees, etc). Other familiar examples of flows are free convection (updrafts above fields, downdrafts above lakes, but also the large scale atmospheric circulation).

In astrophysics the flow particles are not always neutral (molecules, atoms), but can be partially or fully ionized (electrons, ions, possibly mixed with neutrals). The forces acting on such particles are then not only pressure and gravity, but also the electromagnetic (or Lorentz) force.

Before we begin to describe fluid dynamical phenomena in astrophysics, let us first discuss the various astrophysical phenomena we see in the sky in a clear night. Not all of them will directly be relevant to fluid dynamics, at least not in the narrower sense. Also, some phenomena would rather be classified as geophysical fluid dynamics, but again, from a broader perspective they can also be relevant to astrophysics.

3 Examples of astrofluids in the sky

Let us think of some typical phenomena that everybody can observe, either with the naked eye or with a small telescope. We order them according to increasing distance from the Earth.

1. The aurora, or Northern lights, can be seen regularly in northern latitudes, but sometimes also further south.

2. The Sun, sunspots, 11-year cycle, prominences. In order to see some of those phenomena a little telescope or binoculars would help. Some big sunspot groups can sometimes also be seen with the naked eye at sunset when the Sun is not too bright. Also seen can be traces of the solar granulation, a convective flow pattern on the solar surface, but here one definitely needs a small telescope.

3. Planets, Saturn’s rings, Shoemaker-Levy crash onto Jupiter, Jupiter’s red spot are some topics related to our solar system. The following questions arise: how were the planets formed, what do Saturn’s (and other giant planet’s) rings tell us, why was it so difficult to predict the consequences of the comet crash on Jupiter, or what kept Jupiter’s giant red spot together for so long?
4. The zodiacal light is a relic of the protostellar disc that surrounded the forming Sun and out of which planets formed. The zodiacal light can best be seen in the morning hours of the months October/November and the evening hours of February/March, when the ecliptic stands most steeply above the horizon.

5. Meteors, meteorites, and comets also tell us a lot about the solar system. Some meteorites are magnetized, which provides evidence that the early protostellar disc was magnetized when it cooled down below the Curie point, where ferromagnetic materials are able to hold a magnetic field. One particular meteorite has also been used as evidence of life on Mars, but this story now turned out to be a red herring.

6. The Sun and other stars, variable stars, binaries, supernovae (SN), supernova remnants (SNR) are examples in astrophysics where hydrodynamics and magnetic fields in stars play important roles. Stars may show complicated motions (circulation, granulation, oscillations), which can sometimes be very violent (supernovae). In other cases, they lead to magnetic field generation (the solar dynamo), which is the basis of solar and stellar activity, the sunspot cycle, prominences, and so on.

7. The Milky Way, i.e., our Galaxy, as well as other galaxies, are examples of astrophysical bodies on a much larger scale (ten orders of magnitude larger than the Sun). A lot of new physics is introduced by them which explains for example why galaxies can have a spiral structure, or why galaxies can have a halo of synchrotron emission. Many galaxies also show magnetic fields with a large scale pattern, reviving again the interest in dynamo theories. The Andromeda nebula is the brightest spiral galaxy visible with the naked eye.

8. Interstellar clouds, H\textsubscript{II} nebulae, and reflection nebulae are topics of galactic physics which touch to some extent upon questions of hydrodynamics: how do interstellar clouds form (this in turn is related to turbulence in the interstellar medium), and how can stars form out of clouds. H\textsubscript{II} nebulae are best seen on photographs using even an ordinary 50-mm lens. Exposure times of 5 min can be sufficient. The Orion nebula is the most dramatic one an can be seen with the naked eye.

9. Open and globular clusters are also members of our Galaxy which can be seen with the naked eye. With a somewhat broader understanding of what fluid dynamics is, they too belong to the subject. Clusters are selfgravitating bodies consisting of many stars. Similar to an ensemble of randomly moving gas particles in a star, the ensemble of stars too can provide some kind of a pressure that prevents the cluster from collapsing. The Pleiades are an example of a big nearby open cluster. M13 in the constellation of Hercules is a large globular cluster.

10. Cosmology and the early universe are unbounded environments that are often modeled using periodic box simulations. After the electroweak phase transition, when the electromagnetic and weak forces became independent of each other, the universe was highly conducting and highly collisional. It is therefore a perfect environment for applying magnetohydrodynamics (MHD) and turbulence theory. The speed of sound was close to the speed of light, so those flows are rarely relativistic. However, extra effects resulting from the chirality of fermions (the chiral magnetic effect) can play a role and can lead to spontaneous magnetic field generation. Also gravitational waves can be generated. They can potationally be observed with LISA, the Laser Interferometer Space Antenna.

Table 1 gives an overview of the topics mentioned above, and which will be (or should have been) addressed in this course.
Table 1: Summary of topics

<table>
<thead>
<tr>
<th>Material</th>
<th>Planets,</th>
<th>Stars,</th>
<th>Galaxies, AGNs, Clusters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large scale flows</td>
<td>iron, lava</td>
<td>hydrogen, helium</td>
<td>gas, stars</td>
</tr>
<tr>
<td>Discs, winds, jets</td>
<td>differential rotation</td>
<td>rotation and circulation</td>
<td>Keplerian flow, spirals</td>
</tr>
<tr>
<td>Violent events</td>
<td>protoplanetary</td>
<td>around compact objects</td>
<td>AGNs, Quasars</td>
</tr>
<tr>
<td>Flows &amp; turbulence</td>
<td>circulation</td>
<td>convection</td>
<td>galactic rotation, SN explosions</td>
</tr>
<tr>
<td>Magnetic fields</td>
<td>compass, space crafts</td>
<td>cyclic fields</td>
<td>synchrotron emission</td>
</tr>
</tbody>
</table>

Basic equations

4 The equation of motion

At the basis of mechanics lies the *equation of motion* describing the motion of a single particle or an ensemble of particles (and a gas is an ensemble of molecular or atomic particles) under the influence of external forces. The equation of motion is also known as Newton’s first law and can be written as

\[ m \frac{du}{dt} = F, \quad (1) \]

where \( d/dt \) denotes time differentiation, \( v \) is the velocity vector and \( F \) is the force acting on the particle of mass \( m \). Table 2 gives examples of forces.

Table 2: Table of forces that can govern the motion of a single particle of mass \( m \), electric charge \( q \), radius \( r \), volume \( V \), in the presence of gravity \( g \), a magnetic field \( B \), overall rotation \( \Omega \), in a medium of dynamical viscosity \( \mu \), and density \( \varrho \).

<table>
<thead>
<tr>
<th>Name</th>
<th>symbol</th>
<th>expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>gravity force</td>
<td>( F_g )</td>
<td>( mg )</td>
</tr>
<tr>
<td>electrostatic force</td>
<td>( F_{el} )</td>
<td>( qE )</td>
</tr>
<tr>
<td>Lorentz force</td>
<td>( F_L )</td>
<td>( qv \times B )</td>
</tr>
<tr>
<td>Coriolis force</td>
<td>( F_{Cor} )</td>
<td>( -m\Omega \times v )</td>
</tr>
<tr>
<td>centrifugal force</td>
<td>( F_c )</td>
<td>( m\Omega^2r )</td>
</tr>
<tr>
<td>Stokes drag force</td>
<td>( F_D )</td>
<td>( -6\pi\mu rv )</td>
</tr>
<tr>
<td>turbulent drag force</td>
<td>( F_D^{(turb)} )</td>
<td>( -C_D\pi r^2 g\varrho^2 \hat{v} )</td>
</tr>
<tr>
<td>buoyancy force</td>
<td>( F_{buoy} )</td>
<td>( \Delta \varrho V g )</td>
</tr>
</tbody>
</table>

5 Text books

Some useful text books include:


The book by Zeilik has many pictures and explains things qualitatively (no formulae). The book by Shore is more mathematical and interesting to read. It has many misprints however. Finally, the book by Shu covers things in much more detail. Regarding accretion discs good books are those by Frank et al (1992) and Campbell (1997):
5.1 The buoyancy force: the hot air balloon as an example

Gas motions in the Earth’s and solar atmospheres are often driven by buoyancy. As an example we now calculate the buoyancy force from a hot air balloon. The buoyancy force results from a lower density inside the balloon (or any other container) and the density outside it. So the buoyancy force is given by

\[ F_{\text{buoy}} = -\Delta \rho V g, \]  

(2)

where \( g \) is the gravitational acceleration (\( \approx 10 \text{ m s}^{-2} \)), and \( \Delta \rho = \rho_i - \rho_e \) is the density difference between the interior and the exterior of the balloon. If the density within the balloon is smaller than outside, \( \Delta \rho < 0 \) and \( F_{\text{buoy}} \) is positive. The density deficit depends on the temperature excess and, assuming that there is no pressure difference (which is justified for a hot air balloon), the two are proportional to each other, so

\[ \frac{\Delta \rho}{\rho} = -\frac{\Delta T}{T}. \]  

(3)

Let us assume that the temperature inside the balloon is 80\(^\circ\)C and the exterior temperature is 20\(^\circ\)C. The temperature difference is then 60\(^\circ\)C = 60 K, and the absolute temperature is then is \( T = 20 \text{ K} + 273 \text{ K} \approx 300 \text{ K} \) (Kelvin), so

\[ \frac{\Delta \rho}{\rho} = -\frac{60}{300} \approx 0.2. \]  

(4)

So, one cubic meter of hot air is about 20% lighter than cold air. Since the density of air is approximately \( \rho = 1 \text{ kg m}^{-3} \) we have \( \Delta = 0.2 \text{ kg m}^{-3} \). The larger the balloon, the more hot air there is and the lighter is the balloon. Assuming a spherical shape with radius \( R \) the volume of the balloon is \( V_{\text{balloon}} = \frac{4}{3} \pi R^3 \), so the upward force of the entire balloon is

\[ (-\Delta \rho)gV = \frac{4\pi}{3} R^3 (-\Delta \rho)g. \]  

(5)

This has to be balanced against the weight of the balloon, which is \( mg \), if \( m \) is the mass of the payload. Thus, we have

\[ \frac{4\pi}{3} R^3 \Delta \rho = m. \]  

(6)

If we want to know the size of the balloon necessary to carry, say, \( m = 500 \text{ kg} \), we have

\[ R = \left( \frac{3 m}{4\pi \Delta \rho} \right)^{1/3} \approx \left( \frac{1}{4} \frac{500 \text{ kg}}{0.2 \text{ kg m}^{-3}} \right)^{1/3} = \sqrt[3]{625} \text{ m} \approx 9 \text{ m}, \]  

(7)

which seems quite plausible.

5.2 The perfect gas. Equation of state

We need an equation of state that relates the pressure \( p \) of a gas to its density \( \rho \) and its temperature \( T \). For a perfect gas this relation is

\[ p = \frac{\mathcal{R}}{\mu} T \rho, \]  

(8)

where \( \mathcal{R} = 8315 \text{ m}^2 \text{s}^{-2} \text{K}^{-1} \) (this is a script or curly \( \mathcal{R} \)) is the universal gas constant and \( \mu \) is the molecular weight (dimensionless), which is the atomic or molecular mass expressed in units of 1 amu (to a good approximation, \( \mu_{\text{H}} \approx 1 \)). The quantity \( \mathcal{R}T/\mu \) has the dimensions of a velocity squared. As we will
see later, this quantity equals the square of the sound speed in a situation where changes in the pressure and density are isothermal\(^1\).

The quantity

\[ c_{s}^{(\text{isoth})} = \left( \frac{RT}{\mu} \right)^{1/2} \tag{9} \]

is therefore also referred to as the *isothermal* sound speed. For air the value of \( \mu \) is 28.8, so

\[ c_{s}^{(\text{isoth})} \approx \left( \frac{8315 \times 300}{28.8} \right)^{1/2} \text{m/s} \approx 300 \text{ m/s} \tag{10} \]

For ionised hydrogen \( \mu = 0.5 \) (the atomic mass is 1 and the number of particles 2, because there are protons and electrons). However, in the Sun, as well as elsewhere in the cosmos, there is also helium and the value of \( \mu \) is then around 0.6. On the other hand, the presence of neutral and molecular hydrogen increases the average value. Approximate values (to an order of magnitude) of \( c_{s}^{(\text{isoth})} \) and \( T \) are given in Table 3.

<table>
<thead>
<tr>
<th>( T )</th>
<th>( c_{s}^{(\text{isoth})} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^2 ) K</td>
<td>1 km/s</td>
</tr>
<tr>
<td>( 10^4 ) K</td>
<td>10 km/s</td>
</tr>
<tr>
<td>( 10^6 ) K</td>
<td>100 km/s</td>
</tr>
</tbody>
</table>

**Specific heats**: The energy content of a gas is measured by its specific heat, which is the energy needed to increase the temperature by one degree. This quantity can be measured by holding either the volume of the gas constant (specific heat at constant volume, \( c_v \)) or by keeping the pressure constant (specific heat at constant pressure, \( c_p \)). In general, the specific heat at constant volume is smaller than the specific heat at constant pressure, because when the pressure is constant the energy is not only used to increase the temperature, but also to increase the volume. The work associated with this is \( p \Delta V = R/\mu \Delta T \), and therefore

\[ c_p - c_v = \frac{p \Delta V}{\Delta T} = \frac{R}{\mu} \tag{11} \]

According to the kinetic theory of gases (i.e., the theory that describes the gas as noninteracting particles) the specific heat at constant volume is equal to \( R/(2\mu) \) times the number of degrees of freedom \( f \) of a single particle (atom or molecule), i.e., \( c_v = fR/(2\mu) \). Because of Equation (11), we have \( c_p = (f + 2)R/(2\mu) \). Therefore the ratio of the two specific heats, \( \gamma \equiv c_p/c_v \), is equal to

\[ \gamma = \frac{f + 2}{f}. \tag{12} \]

For a mono-atomic gas \( f = 3 \), corresponding to the three directions of translation, so \( \gamma = 5/3 = 1.67 \). Further, for a bi-atomic (dumbbell-like) molecule there are two additional degrees of freedom corresponding to the rotation of the molecule about the axis connecting the two atoms and perpendicular to it, so \( f = 3 + 2 \) and \( \gamma = 7/5 = 1.4 \). The third rotation axis is only distinguished in molecules with more than two atoms. So, for example in CO\(_2\) \( f = 6 \) and therefore \( \gamma = 8/6 = 4/3 \approx 1.33 \). Yet, values of \( \gamma \) closer to unity are possible when the molecules exhibit various kinds of oscillations that further increase the number of degrees of freedom.

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\(^1\) When the changes are adiabatic, e.g., when thermal conduction is weak, the sound speed is slightly larger: \( c_s = \sqrt{\gamma c_{s}^{(\text{isoth})}} \).
5.3 The isothermal atmosphere

Things are changing as we rise. The exterior density decreases, decreasing therefore the buoyancy force. On the other hand, the pressure decreases, so the balloon (or gas parcel) expands and so the interior density also decreases. Which one decreases faster, depends on the temperature profile in the atmosphere. The simplest type of atmosphere is the isothermal atmosphere, ie one where the temperature is constant.

In any atmosphere in hydrostatic (or mechanical) equilibrium the weight (per unit area) of a thin layer of gas, \( \rho gdz \), increases the pressure by the amount \( p_{\text{top}} - p_{\text{bot}} = -\Delta p = \rho gdz \), so the condition of hydrostatic equilibrium is

\[
\frac{dp}{dz} = -\rho g, \quad (13)
\]

where \( g \) is the gravitational acceleration (\( \approx 10 \text{ m/s}^2 \) for the Earth and \( \approx 300 \text{ m/s}^2 \) at the solar surface).

If the atmosphere is isothermal, then \( p = c_s^2 \rho \), where \( c_s^2 = \text{const.} \) In that case we obtain

\[
\frac{d}{dz}(c_s^2 \rho) = c_s^2 \frac{d\rho}{dz} = -\rho g \quad (14)
\]

or (after dividing by \( \rho \) and \( c_s^2 \))

\[
\frac{1}{\rho} \frac{d\rho}{dz} = \frac{d \ln \rho}{dz} = -g/c_s^2 = \text{const},
\]

so

\[
\ln \rho = -gz/c_s^2 + \ln \rho_0, \quad (16)
\]

where \( \ln \rho_0 \) is an integration constant.

So, \( \rho \) decreases exponentially with height, ie

\[
\rho = \rho_0 e^{-z/H}, \quad (17)
\]

where

\[
H = c_s^2 / g, \quad (18)
\]

is also called the scale height of the atmosphere.

Furthermore, since \( p = c_s^2 \rho \) we have

\[
p = p_0 e^{-z/H}, \quad (19)
\]

where \( p_0 = c_s^2 \rho_0 \).

5.4 Adiabatic changes. Entropy

If a fluid parcel preserves its heat content, ie if radiative losses or other heating mechanisms are unimportant, pressure and density changes are said to be adiabatic. This is described by a quantity called the entropy which is then unchanged. For a perfect gas, we define the specific entropy (ie entropy per unit mass) as

\[
s = c_v \ln p - c_p \ln \rho. \quad (20)
\]

(In principle there could be an additive constant \( s_0 \), but we can put it to zero, because only changes in \( s \) matter.) The entropy per unit mass is relevant, because we consider a bubble of a given mass. The specific entropy, in units of \( c_p \), is

\[
s/c_p = \frac{1}{\gamma} \ln p - \ln \rho = \frac{1}{\gamma} \ln \left(p/\rho^\gamma \right), \quad (21)
\]

so if changes in \( p \) and \( \rho \) are adiabatic, ie if \( s = \text{const} \), then

\[
p = e^{s/c_p} \rho^\gamma \quad (22)
\]
or
\[
\frac{p}{p_0} = \left(\frac{\varrho}{\varrho_0}\right)\gamma
\]  \hfill (23)

In order to understand the evolution of a parcel in an atmosphere it is convenient to compute the vertical dependence of \(s\). For an isothermal atmosphere the vertical gradient is
\[
\frac{1}{c_p} \frac{ds}{dz} = -\frac{1}{\gamma H} + \frac{1}{H} = \frac{\gamma - 1}{\gamma} \frac{1}{H} > 0,
\]  \hfill (24)

so \(s\) increases with height. This means that a rising fluid parcel, whose entropy is conserved, will end up in a location where the surrounding entropy is higher. At the same time, however, the pressure inside and outside the bubble will be the same, so \(p_i - p_e = \Delta p = 0\), where subscripts \(i\) and \(e\) refer to interior and exterior values. Thus, from Equation (21) we have
\[
\Delta s / c_p = -\Delta \ln \varrho.
\]  \hfill (25)

So, since the rising fluid parcel ends up in a location of higher exterior entropy, we have \(\Delta s < 0\) and therefore \(\Delta \ln \varrho > 0\), so the fluid parcel becomes *heavier* and will be pulled back by the gravity. This provides a restoring force proportional to the vertical displacement, which leads to

6 Basic set of equations

6.1 Continuity equation
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]  \hfill (26)

It can be written in various other forms, such as \(D \ln \rho / Dt = -\nabla \cdot \mathbf{u}\).

6.2 Navier-Stokes equation
\[
\rho \frac{D \mathbf{u}}{D t} = -\nabla p - \rho \nabla \Phi + \mathbf{J} \times \mathbf{B} + \nabla \cdot \mathbf{\tau}
\]  \hfill (27)

where \(\Phi\) is the gravitational potential, which obeys \(\nabla^2 \Phi = 4\pi G \rho\). The solution for a sphere, for example, is \(\Phi = -GM/r\), while for a plane layer it is \(\Phi = zg\), up to an additive constant.

For a monatomic gas, the stress tensor is \(\mathbf{\tau} = 2 \rho \nu \mathbf{S}\) with \(S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3} \delta_{ij} \nabla \cdot \mathbf{u}\). In that case, \(\nabla \cdot \mathbf{\tau} = \rho \nu (\nabla^2 \mathbf{u} + \frac{1}{3} \nabla \nabla \cdot \mathbf{u} + 2 \mathbf{S} \nabla \ln \rho \nu)\).

- Note that the \(\nabla \Phi\) term comes with a minus and an additional \(\rho\) factor.
- All the other terms come with no \(\rho\) factor.
- It is sometimes useful to note that \(\rho^{-1} \nabla p = \nabla h - T \nabla s\), where \(h\) is the specific enthalpy. For an ideal gas we have \(h = c_p T\).

6.3 Energy equation
\[
\rho T \frac{D s}{D t} = -\nabla \cdot \mathbf{F}_{\text{rad}} + \mathbf{\tau} : \nabla \mathbf{u} + J^2 / \sigma + \text{nucl. fusion}
\]  \hfill (28)

The \(\mathbf{\tau} : \nabla \mathbf{u}\) is supposed to mean \(\tau_{ij} u_{i,j}\). This is the viscous heating term and it is *positive definite*. In the homework, we are supposed to find that for \(\mathbf{\tau} = 2 \rho \nu \mathbf{S}\) we have \(\mathbf{\tau} : \nabla \mathbf{u} = 2 \rho \nu \mathbf{S}^2\), which is manifestly positive definite. In the optically thick case, the radiative flux is \(\mathbf{F}_{\text{rad}} = -K \nabla T\). The electric conductivity \(\sigma\) can also be written as \(1 / \sigma = \mu_0 \eta\), where \(\mu_0\) is the vacuum permeability and \(\eta\) the magnetic diffusivity.

- In the strictly isentropic case \((s = \text{const})\), we have \(p \propto \rho^\gamma\). However, viscous heating can usually not be neglected!
• The lhs of Equation (28) can also be written as $\rho D e / D t + p \nabla \cdot \mathbf{u}$.

• For an ideal gas, we have $e = c_v T$ and can write the lhs also as $\rho c_v D T / D t + p \nabla \cdot \mathbf{u}$.

• Again for an ideal gas, we can write the lhs as $\rho c_p D T / D t - \nabla p / D t$.

Recall that $\mathcal{R}/\mu = c_p - c_v$, where $\mathcal{R} = k_B/m_u$ is the universal gas constant\(^2\) and $\mu$ is the mean molecular weight (=1 for neutral hydrogen, $\approx$ 1.2 for a neutral hydrogen-helium mixture, 0.6 for an ionized hydrogen-helium mixture).

### 6.4 Induction equation

\[
\frac{\partial B}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - J/\sigma) \tag{29}
\]

\(^2\)Here, $k_B = 1.3806505 \times 10^{-16}$ erg K\(^{-1}\) is the Boltzmann constant, $m_u = 1.66053886^{-24}$ g is the atomic mass unit [0.993 times the proton mass], and so $\mathcal{R} = 8.31447 \times 10^7$ erg g\(^{-1}\) K\(^{-1}\). In astrophysics, $\mu$ is dimensionless rather than in g/mol.