Lecture 4: Rayleigh–Bénard problem

Convection plays an important role in geo- and astrophysics. Without it, the would be no turbulence and no magnetic fields in the Sun, for example. This problem has been studied for a long time and goes back to early work of Count Rumford. In a historical account, Brown (1957) mentions in this connection the year 1797,

> In this chatty style, Count Rumford began to describe in 1797 his discovery of the thermal current in fluids. His first observations of convection currents were accidental, as he watched the behavior of the contents of the bore of a large thermometer he was using in an experiment:
>
> I saw the whole mass of the liquid in the tube in a most rapid motion running swiftly in two opposite directions, up and down at the same time. The bulb of the thermometer, which is of copper, had been made two years before I found leisure to begin my experiments, and having been

The decisive statement about ascending currents is made here:

> the thermometer was agitated. . . . On examining the motion of the spirits of wine with a lens, I found that the ascending current occupied the axis of the tube and that it descended by the sides of the tube. On inclining the tube a little, the rising current moved out of the axis and occupied the side of the tube which was uppermost, while the descending currents occupied the whole of the lower side of it.
>
> The Count recognized that he had made a discovery of the first magnitude, and he correctly assigned the thermal motions to the changes in density of the fluid as a result of heating. The hot liquid, being lighter, tended to rise and the heavier liquid, which was cold, fell, and thus the currents could be separated into a continuous motion. To dramatize the experimental observations of these cur-

But the word convection cas coined only much later.

> It was not until 1834 that the well-known William Prout, writing in one of the *Bridgewater Treatises*, suggested:[2]
>
> There is at present no single term in our language employed to denote this mode of propagation of heat; but we venture to propose for that purpose the term *convection* (*convectio*, a carrying or converging), which not only expresses the leading facts, but also accords very well with the two other terms [conduction and radiation].
>
> Even after this name was suggested, it took twenty years for *convection* to find its way to the universal acceptance which the term now enjoys.

A detailed account of the subsequent discoveries that went into our theoretical understanding of this phenomenon is given by Chandrasekhar (1961). After the original experimental work of Bénard (1900) and theoretical work of Rayleigh (1916), the mathematical formulation that is used nowadays in many textbooks goes back to work by Jeffreys (1926). It was also Jeffreys (1930), who generalized the theory to the compressible case, where the temperature gradient is to be replaced by the superadiabatic gradient, which is the famous Schwarzschild (1906) criterion.

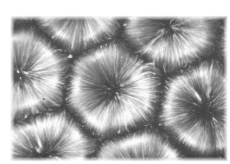

Figure 1: Convection experiment.

Discrepancies between Bénard's original experiments and the theoretical calculations are explained by the importance of surface tension, which was ignored in the early work. This leads to the so-called Marangoni effect which describes the mass transfer along an interface between two fluids due to surface tension gradient.

1 Governing equations

The best studied case is that of very weak stratification, so one can adopt the *Boussinesq approximation*. In the absence of rotation and magnetic field, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{1}$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\frac{1}{\rho_0}\boldsymbol{\nabla}P+\frac{\rho}{\rho_0}\boldsymbol{g}+\nu\nabla^2\boldsymbol{u}, \tag{2}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t}=\kappa\nabla^2T, \tag{3}$$

where $\boldsymbol{g}=(0,0,-g)$ is gravity. The essential point here is that the fluid is incompressible and the density is almost constant except for a weak response to changes in the temperature, which leads to a temperature-dependent buoyancy of the form

$$\rho=\rho_0+\left(\frac{\partial\rho}{\partial T}\right)_P(T-T_0). \tag{4}$$

This coefficient is negative and it is proportional to what is usually called α, i.e., $-\rho_0^{-1}(\partial\rho/\partial T)_P=\alpha$. For a perfect gas, $\alpha=1/T$, but one also wants to consider water and other liquids, where αT is of the order of 10^6; see Chandrasekhar (1961) for details.

The temperature is assumed fixed on the two boundaries, so the hydrostatic solution must obey $\nabla^2T=0$, which leads to a linear temperature profile, $T_0(z)=T_{00}-\beta z$, where β is the negative temperature gradient. The set of linearized equations can then be written in the form

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_1=0 \tag{5}$$

$$\frac{\partial\boldsymbol{u}_1}{\partial t}=-\frac{1}{\rho_0}\boldsymbol{\nabla}P_1+\alpha T_1 g\hat{\boldsymbol{z}}+\nu\nabla^2\boldsymbol{u}_1 \tag{6}$$

$$\frac{\partial T_1}{\partial t}-\beta u_{z1}=\kappa\nabla^2T_1. \tag{7}$$

2 Eliminating the pressure

A common trick is to apply the double curl. With the first curl, we eliminate the pressure, and with the second curl, the buoyancy term has just a z component, i.e.,

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\begin{pmatrix}0\\0\\\alpha gT_1\end{pmatrix}=\boldsymbol{\nabla}\times\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\times\begin{pmatrix}0\\0\\\alpha gT_1\end{pmatrix}=\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\times\begin{pmatrix}+\alpha gT_{1,y}\\-\alpha gT_{1,x}\\0\end{pmatrix}=\alpha g\begin{pmatrix}T_{1,xz}\\T_{1,yz}\\-\nabla_\perp^2T_1\end{pmatrix}, \tag{8}$$

where $\nabla_\perp^2=\partial_x^2+\partial_y^2$ is the horizontal Laplacian and commas denote partial derivatives. Furthermore, making use of the relation $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times=-\nabla^2+\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot$ and the fact that $\boldsymbol{\nabla}\cdot\boldsymbol{u}_1=0$, we have $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u}_1=-\nabla^2\boldsymbol{u}_1$, and so we arrive directly at

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2\boldsymbol{u}_1=\alpha g\hat{\boldsymbol{z}}\nabla_\perp^2T_1, \tag{9}$$

or, more specifically for the z component of the velocity, at

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2u_{z1}=\alpha g\nabla_\perp^2T_1, \tag{10}$$

and for the temperature we have analogously

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)T_1=\beta u_{z1}. \tag{11}$$

This is a system of two partial differential equations that can easily be combined into one:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\nabla^2 u_{z1} = \alpha\beta g \nabla_\perp^2 u_{z1}. \tag{12}$$

(This equation applies equally to T_1.) Furthermore, it is advantageous to nondimensionalize the equations by measuring length in units of d, which is the vertical thickness of the layer, so we write $[x] = d$, and time in units of the viscous time, $[t] = d^2/\nu$.

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\nu}{\kappa}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 u_{z1} = \frac{\alpha\beta g d^4}{\nu\kappa}\nabla_\perp^2 u_{z1}. \tag{13}$$

As a consequence of nondimensionalization, two nondimensional numbers have appeared, the Prandtl and Rayleigh numbers,

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \quad \mathrm{Ra} = \frac{\alpha\beta g d^4}{\nu\kappa}. \tag{14}$$

respectively. Thus, we have

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\mathrm{Pr}\,\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 u_{z1} = \mathrm{Ra}\,\nabla_\perp^2 u_{z1}. \tag{15}$$

3 Spectral ansatz for a specific boundary condition

Before considering the case of general boundary conditions, let us first consider a case where we can assume the solution to be of the form $u_{z1} = \hat{u}_{z1}(z)\, e^{\sigma t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$,

$$\left(\sigma + k^2\right)\left(\mathrm{Pr}\,\sigma + k^2\right)k^2 = \mathrm{Ra}\,k_\perp^2. \tag{16}$$

thus, we have

$$\mathrm{Pr}\,\sigma^2 + (1+\mathrm{Pr})\,\sigma k^2 + k^4 = \mathrm{Ra}\,\frac{k_\perp^2}{k^2}. \tag{17}$$

At this point, we can already say that there exists a nontrivial solution with $\sigma = 0$ when $\mathrm{Ra} = k^6/k_\perp^2$. Such a solution corresponds to the marginally excited, nonoscillatory state.[1]

If $\sigma = 0$ is the critical condition of interest, then the wavevector of interest is that corresponding to the longest wavelength that fits into the domain. Thus now depends on the boundary conditions. In fact, the *only* boundary condition that would be allowed with the adopted spectral ansatz is the stress-free condition, with zero temperature fluctuation, i.e.,

$$u_{1x,z} = u_{1y,z} = u_{1z} = T_1 = 0 \quad (\text{on } z = \pm d/2). \tag{18}$$

In that case, we have

$$u_{1x} \propto \sin k_z z, \quad u_{1y} \propto \sin k_z z, \quad u_{1z} \propto \cos k_z z, \quad T_z \propto \cos k_z z \quad \text{with } k_z = \pi/d. \tag{19}$$

Now that k_z is fixed, we would still be free to vary $k_\perp$. The preferred value of $k_\perp$ is the one that minimizes $\mathrm{Ra}(k_\perp^2) = (k_\perp^2 + k_z^2)^3/k_\perp^2$. Thus we solve $\mathrm{dRa}/\mathrm{d}k_\perp^2 = 0$, i.e.,

$$\mathrm{dRa}/\mathrm{d}k_\perp^2 = 3(k_\perp^2 + k_z^2)^2/k_\perp^2 - (k_\perp^2 + k_z^2)^3/k_\perp^4 = 0. \tag{20}$$

Thus, $3\,k_\perp^2 - (k_\perp^2 + k_z^2) = 0$, and therefore

$$k_\perp^2 = k_z^2/2 = \pi^2/2, \tag{21}$$

or $k_\perp = \pi/\sqrt{2} \approx 2.22$; see Figure 2.

For $k_\perp = \pi/\sqrt{2}$ and $k_z = \pi$, the value of Ra is

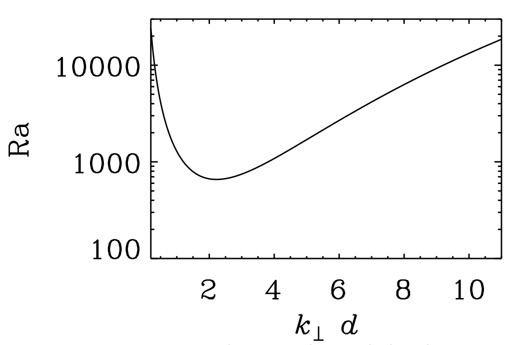

Figure 2: $\mathrm{Ra}(k_\perp^2) = (k_\perp^2 + k_z^2)^3/k_\perp^2$.

$$\mathrm{Ra} = (k_\perp^2 + k_z^2)^3/k_\perp^2 = (\pi^2/2 + \pi^2)^3/(\pi^2/2) = \frac{27}{4}\pi^4 \approx 657.5\,. \tag{22}$$

which is the *critical* value for the onset of convection in a horizontally infinitely extended layer. Obviously, if one does a simulation in a horizontally periodic domain, it won't be infinitely extended, and therefore $k_\perp$ can only be a multiple of the smallest horizontal domain size.

[1] There could in principle also be another condition for a marginally excited solution where $\mathrm{Re}\,\sigma = 0$, but $\mathrm{Im}\,\sigma \neq 0$, which would be oscillatory and thus with nonvanishing σ.

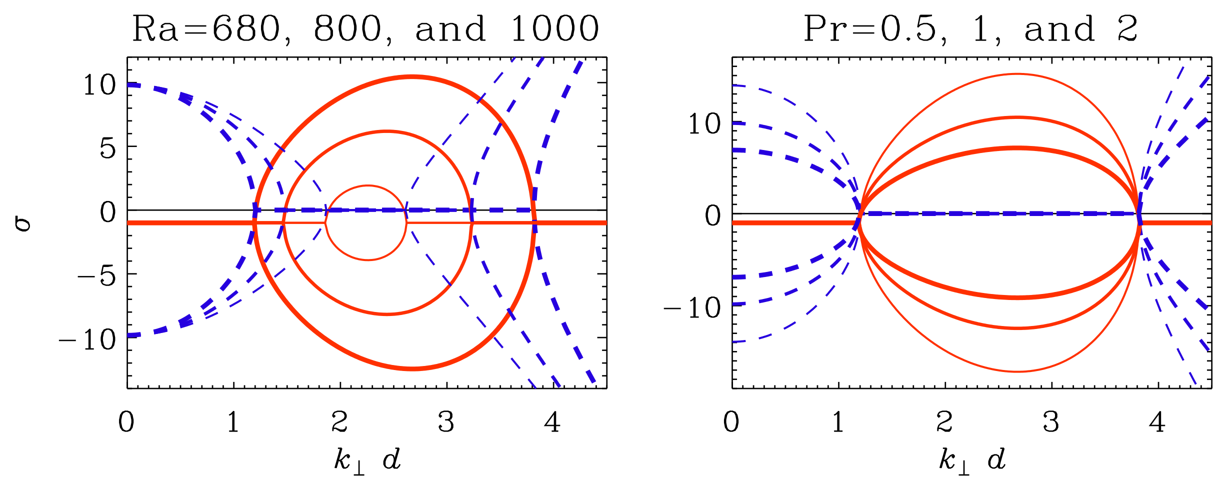

Figure 3: Dispersion relation for different values of Ra and Pr; increasing values are indicated by increasing thickness.

4 Dispersion relation

Note that have not yet solved the dispersion relation. Let us now solve Equation (17) for fixed $k_z = \pi$ as a function of $k_\perp^2$. This, we have $k^2 = k_\perp^2 + \pi^2$. Let us rewrite Equation (17) in standard form as

$$\sigma^2 + \frac{1+\mathrm{Pr}}{\mathrm{Pr}}\,\sigma k^2 + \frac{k^4}{\mathrm{Pr}} - \frac{\mathrm{Ra}}{\mathrm{Pr}}\,\frac{k_\perp^2}{k^2} = 0. \tag{23}$$

and write the solutions as

$$\sigma_\pm = -\frac{1+\mathrm{Pr}}{2\,\mathrm{Pr}}k^2 \pm \sqrt{\frac{(1+\mathrm{Pr})^2}{4\,\mathrm{Pr}^2}k^4 - \frac{k^4}{\mathrm{Pr}} + \frac{\mathrm{Ra}}{\mathrm{Pr}}\,\frac{k_\perp^2}{k^2}}. \tag{24}$$

The result is plotted in Figure 3.

References

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