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The dependence of the viscosity in accretion discs on the shear/vorticity ratio

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ABSTRACT

We estimate the Shakura–Sunyaev viscosity parameter α for different values of the shear/vorticity ratio, σ/ω, using local simulations of dynamo-generated turbulence. We find that the time average of α is approximately proportional to σ/ω (at least for $\sigma/\omega<10$). We point out that this result may have important implications for the properties of thick accretion discs, because there ω is small and α would then tend to be large. Our result may also be important for accretion discs around black holes, because σ/ω becomes large in the inner 10 Schwarzschild radii as a result of relativistic effects.

Key words: accretion, accretion discs – black hole physics – turbulence.

1 INTRODUCTION

There is now strong evidence that the turbulent viscosity in accretion discs is caused by a magnetic instability (Balbus & Hawley 1991). Three-dimensional local turbulence models are now available that have enabled us for the first time to 'measure' the normalized disc viscosity α, which is the turbulent viscosity divided by the sound speed c_s and the scale-height H. Previously, α was often taken as a free parameter in global models. However, we are now in a position to determine the dependence of α on local properties of the disc.

It is plausible that the strengths of large-scale shear and vorticity play important roles in determining the magnitude of α. In accretion discs the angular velocity Ω varies with radius like $\Omega \sim r^{-q}$, where $q=3/2$ for Keplerian rotation. The growth rate of the Balbus–Hawley instability is proportional to q (Balbus & Hawley 1992). Furthermore, for $q\rightarrow 2$ the disc approaches a state of constant angular momentum and becomes unstable by Rayleigh's criterion. Therefore we expect α to increase with q. Indeed, there are several circumstances where $q>3/2$, especially in the inner parts of accretion discs, as we shall see in Section 3.

However, q is a coordinate-dependent quantity and therefore not useful in more general circumstances, for example near black holes, where the coordinate r has no dynamical significance. Therefore we shall express α in terms of shear and vorticity of the background rotation.

Since the magnetic shear instability is local, it is possible to estimate the viscosity using a local model either with an externally applied magnetic field (Hawley, Gammie & Balbus 1995; Matsumoto & Tajima 1995) or without (Brandenburg et al. 1995). Here we consider only the latter case, where a magnetic field is generated self-consistently by dynamo action. This removes the ambiguity arising from the otherwise ill-defined magnetic field strength. This is the relevant case for discs where the central object has no, or only a weak, magnetic field, and where the field is generated entirely within the disc on the scales resolved in the simulation.

In a local model all radial gradients are ignored except for the angular velocity. This is an important approximation that becomes invalid near the central object. Effects owing to the converging accretion flow, for example, are ignored. When we talk about effects owing to deviations from a Keplerian flow near the central object, we have to keep in mind that we refer only to effects that show up locally. We assume therefore that they are the dominant ones.

2 SHEAR/VORTICITY IN A SHEARING BOX

In the shearing box approximation the total velocity is decomposed into three components: rigid rotation, linear shear, and a turbulent component. Near a reference point

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(R, ϕ_0) (in cylindrical polar coordinates) the background velocity (i.e. without the turbulent component) is

$$\boldsymbol{U}=\boldsymbol{\Omega}_0 \times \boldsymbol{r}+\boldsymbol{u}_{\mathrm{shear}}, \tag{1}$$

where $\boldsymbol{\Omega}_0=\hat{\boldsymbol{z}}\Omega_0$ with $\Omega_0=\Omega(R)$ being the angular velocity at the reference radius, and $\boldsymbol{r}=(R+x, y, 0)$ is the cylindrical radius vector in local Cartesian coordinates, $x=r-R$, $y=(\phi-\phi_0)R$, and z. This flow results from the balance between the radial component of gravity, the centrifugal force, the Coriolis force, and possibly an extra body force $\boldsymbol{f}=p\Omega_0^2\hat{\boldsymbol{x}}$. In the local approximation, $x/R \ll 1$, we have

$$-\frac{GM}{R^2}\left(1-2\frac{x}{R}\right)+\Omega_0^2 R\left(1+\frac{x}{R}\right)+2\Omega_0 u_y^{(0)}+p\Omega_0^2 x=0. \tag{2}$$

This yields the value of the basic rotation $\Omega_0^2=GM/R^3$ and the shear flow, $\boldsymbol{u}_{\mathrm{shear}}=\hat{\boldsymbol{y}}u_y^{(0)}(x)$ with

$$u_y^{(0)}(x)=-q\Omega_0 x, \tag{3}$$

where $q=(3+p)/2$. For $p=0$ we have ordinary Keplerian rotation, $q=3/2$. This velocity field may be described by shear and vorticity tensors,

$$\sigma_{ij}=\tfrac{1}{2}(\partial U_i/\partial x_j+\partial U_j/\partial x_i)-\tfrac{1}{3}\delta_{ij}\partial U_k/\partial x_k, \tag{4}$$

$$\omega_{ij}=\tfrac{1}{2}(\partial U_i/\partial x_j-\partial U_j/\partial x_i). \tag{5}$$

In the following we characterize the flow by σ^2 and ω^2, which are coordinate-independent. In our case their moduli are

$$\sigma=|\partial U_x/\partial y+\partial U_y/\partial x|/\sqrt{2}=q\Omega_0/\sqrt{2}, \tag{6}$$

$$\omega=|\partial U_x/\partial y-\partial U_y/\partial x|/\sqrt{2}=(2-q)\Omega_0/\sqrt{2}. \tag{7}$$

Using this background velocity, we computed models similar to those in Brandenburg et al. (1996a). In the present paper we use models for different values of q and compute the turbulent viscosity ν_{t}, which describes the magnitude of the total horizontal stress relative to the large-scale shear, i.e.

$$\langle\rho u_x u_y-B_x B_y/\mu_0\rangle=-\nu_{\mathrm{t}}\langle\rho\rangle r\frac{\partial\Omega}{\partial r}. \tag{8}$$

We then express ν_{t} in units of c_{s} and H, i.e. $\nu_{\mathrm{t}}=\alpha c_{\mathrm{s}}H$, and use $r\partial\Omega/\partial r=-q\Omega$.

The value of α is not constant, but depends on the magnetic field strength which, in turn, varies in a cyclic manner (Brandenburg et al. 1995). Therefore we give here the mean value obtained by averaging over approximately one cycle. This introduces uncertainty, because the cycle length varies somewhat in time and from case to case. In a few cases we have run for up to three magnetic cycles. We should also point out that α increases somewhat with increasing numerical resolution (Brandenburg et al. 1996a). In order to perform a survey in parameter space, we restrict ourselves to a modest resolution of $31\times63\times32$ meshpoints.

The solutions for different values of q have been obtained by restarting the simulation from another snapshot. It takes some time for the solution to settle, but, after less than one orbit, α has adjusted roughly to the new value. For $q\lesssim1$ it becomes increasingly difficult to find a unique value of α because of rather long transients. In this case α seems to depend on the initial conditions and therefore we often considered different initial conditions. For example, in a case with $q=0.5$ the magnetic field was modified such that $\langle B_y\rangle=0$ initially. In this case $\langle B_y\rangle$ began to evolve away from zero in an oscillatory manner, but α was roughly similar to the values found before. However, at much later times (more than 100 orbits), α became much smaller. We are not sure whether the field was very slowly decaying, or whether there is possibly a dynamo even in the limit of vanishing shear, driven possibly by the effects of buoyancy (Różyczka, Turner & Bodenheimer 1995). However, for small negative values of q ($q=-0.1$) we found that α decayed to zero very quickly.

The results are summarized in Table 1. Note that α increases monotonically with q. The values of α are well represented by a linear dependence on the ratio σ/ω,

$$\alpha=\alpha_0\frac{\sigma}{\omega} \tag{9}$$

(see Fig. 1). The case of Keplerian rotation, $q=3/2$, corresponds to $\sigma/\omega=3$. From the simulations we find $\alpha_0\approx0.0015$. This simple linear dependence, obtained here from a fit to the results of fully three-dimensional simulations, is quite

Table 1. Summary of the runs. In addition to the value of α and its rms value, the normalized Maxwell stress, the inverse plasma beta, and the ratio of magnetic to turbulent kinetic energies are also given. For some values of σ/ω there are two different results corresponding to runs with different initial conditions.

q	0.1	0.5	1.0	1.5	1.7	1.8
σ/ω	0.05	0.3	1.0	3.0	5.7	9.0
α		.0001	.0015	.005	.008	.013
	.0001	.0010	.0033			
rms(α)	.0001	.0009	.0015	.003	.003	.003
$\langle B_xB_y\rangle/\langle\mathbf{B}^2\rangle$		.093	.096	.087	.098	.100
	0.012	.076	.089			
$\langle\mathbf{B}^2\rangle/\langle 8\pi p\rangle$		.0003	.011	.049	.065	.112
	0.008	.013	.028			
$\langle\mathbf{B}^2\rangle/\langle\rho\mathbf{u}^2\rangle$		2.2	3.5	4.4	2.2	1.7
	12.5	6.5	5.0			

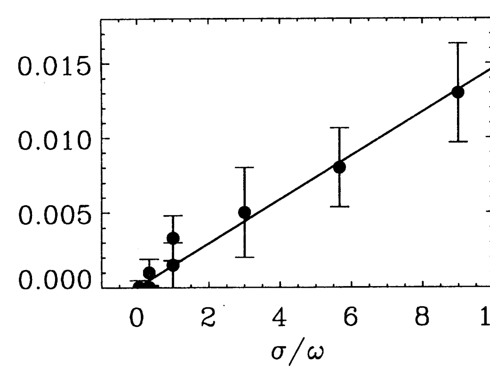

Figure 1. α as a function of σ/ω. The vertical bars give the rms value of α. The solid line is a linear fit through the data. The uncertainty in the averages is generally *less* than the rms value.

striking, and one wonders whether it could be understood in terms of some basic physics. The fact that α increases with σ/ω suggests that shear enhances the turbulence, or makes it at least more effective in transporting angular momentum. Using general arguments based on comparison of relevant length-scales, Godon (1995) also found that α should depend on the shear.

Equation (9) would imply an infinitely large value of α for discs of constant angular momentum ($\omega=0$). It is quite plausible that the linear dependence implied by equation (9) will break down as $\sigma/\omega\rightarrow\infty$. Indeed, as we approached $q\rightarrow 2$ the strength of the turbulence increased substantially and $q=1.8$ was the largest value for which we still obtained a statistically steady state. This is somewhat surprising, because the equations do not immediately suggest that anything peculiar should happen in this case. However, for $q\geq 2$ the system becomes unstable by Rayleigh's criterion and it is not clear whether there is a mechanism that would lead to saturation.

In the following we discuss possible implications of our fit (9). We discuss in particular the case $q>3/2$, where the uncertainty in the linear fit is much smaller than for smaller values of q.

3 APPLICATIONS

The dimensionless viscosity coefficient α is defined by the equation $\nu_t=\alpha c_s H$, where ν_t is the turbulent kinematic viscosity coefficient, c_s is the local sound speed, and $H(\sim c_s/\Omega)$ is the local pressure scaleheight. In this formula (together with equation 9), all the quantities on the right-hand side are defined locally by coordinate-independent quantities. We suggest therefore that this formula may be applicable for a much wider range of hydrodynamical calculations than our local models.

We discuss two different cases where σ/ω can be larger than 3, the asymptotic value for Keplerian rotation. The first concerns thick accretion discs, where additional pressure support can make the rotation profile steeper. Another important case comprises the inner parts of thin accretion discs, where, even for Keplerian rotation, $\sigma/\omega\gg 3$.

3.1 Thick accretion discs

Thick accretion discs were proposed in the early 1980s (see Abramowicz, Calvani & Nobili 1980) as possible models for quasars and other active galactic nuclei. One of the particularly characteristic features of the thick discs is the presence of long and narrow funnels along the axis of rotation. It is believed that these funnels may play a crucial role in the collimation of relativistic jets emerging from quasars.

Thick accretion discs have been mostly studied in the case where the angular momentum distribution is uniform and the vorticity zero. Such discs are possible only for very *small* values of α, because otherwise viscosity would lead to a rapid redistribution of angular momentum, resulting in a state where angular momentum increases outwards. This scenario is now in conflict with equation (9), because for $\omega\rightarrow 0$ our formula predicts *large* viscosity, in contrast to the original assumption. We therefore expect that a subsequent redistribution of angular momentum leads to values of α small enough that the angular momentum distribution is no longer affected. The value to which α will relax may determine the shape of the flow.

Using two-dimensional time-dependent hydrodynamical simulations, Igumenshchev, Chen & Abramowicz (1996) have demonstrated, for two different values of α (10^{-2} and 10^{-3}), that the funnels do not disappear when the angular momentum distribution changes during the viscous evoluton. It is not known what happens if the value of α is much larger, and strongly varies in the flow. It would be interesting to see whether this leads to some steady 'canonical' angular momentum distribution, and whether the empty funnels along the axis of rotation survive the rapid viscous evolution close to the axis. Recent calculations by Chen, Abramowicz & Lasota (1996) and Narayan, Honma & Kato (1996), based on stationary models employing Newtonian formulae in the vertical direction and pseudo-Newtonian formulae in the radial one, seem to indicate that, when α is constant and >0.1 everywhere, the funnels do not survive (Narayan et al. 1996).

3.2 Thin and slim accretion discs around black holes

For the Schwarzschild (non-rotating) black hole with mass M and gravitational radius $r_G=2GM/c^2$, the Keplerian angular velocity of rotation in terms of the Schwarzschild radial coordinate r is $\Omega=GMr^{-3/2}$, and the Keplerian angular momentum is $\ell=\Omega\mathscr{R}^2$, with $\mathscr{R}^2=r^2/(1-r_G/r)$ being the radius of gyration. From these formulae one deduces that the Keplerian angular momentum is not a monotonic function of radius, but it has a minimum at $r=3r_G$. For $r<3r_G$ no stable Keplerian orbits are possible. In thin and slim (Abramowicz et al. 1980) accretion discs around Schwarzschild black holes, matter loses its angular momentum as a result of the action of a viscous torque, and spirals down towards the central black hole. In thin accretion discs, for $r>3r_G$ the inward radial component of velocity is much smaller than the azimuthal component, which is almost Keplerian. For $r<3r_G$, both components are comparable because the matter is radially free-falling. Thus the general distribution of the angular momentum in the standard disc corresponds to almost Keplerian for $r>3r_G$, and almost constant for $r<3r_G$. Somewhere in the range $2r_G<r<3r_G$ the character of the flow changes, and for this reason this location is often called the inner edge of the disc.

Relativistic expressions for the shear and vorticity for a general rotational motion in a static space–time read

$$\sigma^2=-\tfrac{1}{4}(1-\Omega\ell)^{-2}\mathscr{R}^2(\nabla_i\Omega)(\nabla^i\Omega), \tag{10}$$

$$\omega^2=-\tfrac{1}{4}(1-\Omega\ell)^{-2}\mathscr{R}^{-2}(\nabla_i\ell)(\nabla^i\ell) \tag{11}$$

(see Abramowicz 1982). From this we derive the result that, in the case of Keplerian rotation around a Schwarzschild black hole, the shear/vorticity ratio equals

$$\frac{\sigma}{\omega}=3\,\frac{r-r_G}{r-3r_G}. \tag{12}$$

This is shown in Fig. 2 (by the heavy line) as a function of the radial coordinate r. For comparison, the shear/vorticity ratio for Keplerian rotation of the pseudo-Newtonian potential is shown by the dotted line. The pseudo-

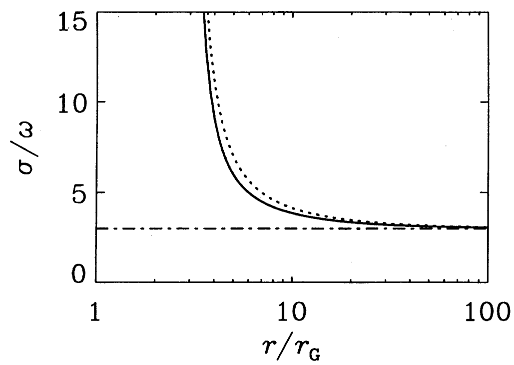

Figure 2. σ/ω as a function of r for a Schwarzschild black hole. The solid line is the exact expression (12) and the dotted line is the result obtained using the pseudo-Newtonian potential. The dot-dashed line marks the asymptotic value.

Newtonian potential was proposed by Paczyński & Wiita (1980) as the Newtonian model for a non-rotating black hole, and found to be an excellent approximation in most hydrodynamical applications (see e.g. Artemova, Björnsson & Novikov 1996).

In theoretical studies of accretion discs, it is customary to assume that α is constant through the disc, or that it varies as a certain power of H/r, when H is the vertical pressure scaleheight (or the vertical half-thickness of the disc). Our viscosity formula implies, however, that the value of $\alpha=\alpha_0\sigma/\omega$ strongly increases towards the inner edge of the disc, $r\approx 3r_G$. The region affected by this is not just the innermost part of the disc. Asymptotically, far from the central black hole, i.e. for $r\gg r_G$, one has for strictly Keplerian rotation $\sigma/\omega=3$, and $\alpha=\alpha_0\approx 0.005$. At $r=10r_G$ for strictly Keplerian flow one has $\sigma/\omega=4.1$ and $\alpha=0.007$, and, for $r=5r_G$, one has $\sigma/\omega=7.0$ and $\alpha=0.012$. Note that the disc radiates most of the energy from the region $10r_G>r>3r_G$ and thus a knowledge of the precise value of α there is of direct observational interest.

For slim accretion discs and optically thin advection-dominated flows (see Abramowicz 1996 and Narayan 1996 for recent reviews), the azimuthal motion of the matter is non-Keplerian even for $r>10r_G$. For low values of α ($\lesssim 0.01$) the specific angular momentum in the inner disc is practically constant and matter is accreted via 'relativistic Roche lobe overflow'. For higher values of α the sub-Keplerian specific angular momentum distribution is steeper and matter undergoes viscous accretion. These results are obtained assuming a constant α. As in the case of thick discs, discussed above, and for the same reasons, this behaviour is in contradiction with our results concerning the dependence of viscosity on the shear/vorticity ratio. One may expect that, if this dependence is included in the model, the resulting flow will be different from the one obtained assuming a constant α.

In addition, from theoretical studies it follows that the various *qualitative* properties of the flow depend on whether the value of α is larger or smaller than the critical value. For several of these properties $\alpha_{crit}\approx 0.1$. For example, the sonic point for flows with $\alpha<\alpha_{crit}$ has saddle topology, and that for flows with $\alpha>\alpha_{crit}$ has nodal topology, which relates to the presence of the unstable standing acoustic mode (for references see Kato, Abramowicz & Chen 1996). Similarly, the possible existence of a range of radii with a hot, optically thin, thermally overstable accretion flow cooled by advection depends on whether $\alpha<\alpha_{crit}$ (see Abramowicz et al. 1995; Chen et al. 1995, 1996). Our result suggests a rather high value of α, which most probably is greater than the critical value in both cases discussed above.

4 CONCLUSIONS

We have demonstrated, using numerical simulations, that the α-viscosity arising from dynamo-generated turbulence is proportional to the shear/vorticity ratio. On the one hand, this provides a starting point for two fundamental theoretical questions: (1) why is it so and (2) how do non-linear effects influence the formula in the limit of vanishing vorticity? On the other hand, our formula may be used in a variety of large-scale numerical hydrodynamical simulations, because it uses only coordinate-invariant quantities that could easily be defined and calculated in any particular situation. We should point out, however, that we used the relation (8) in which the stress is proportional to $\langle\rho\rangle$, a property that has to be tested by future calculations (cf. Brandenburg et al. 1996b).

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REFERENCES

Abramowicz M. A., 1982, ApJ, 254, 768
Abramowicz M. A., 1996, in Kato S., Inagati S., Mineshige S., Fokue J., eds, Physics of Accretion Disks. Gordon and Breach, New York, p. 1
Abramowicz M. A., Calvani M., Nobili L., 1980, ApJ, 242, 772
Abramowicz M. A., Chen X., Kato S., Lasota J.-P., Regev O., 1995, ApJ, 438, L37
Artemova I. V., Björnsson G., Novikov I. D., 1996, ApJ, 461, 565
Balbus S. A., Hawley J. F., 1991, ApJ, 376, 214
Balbus S. A., Hawley J. F., 1992, ApJ, 400, 610
Brandenburg A., Nordlund Å., Stein R. F., Torkelsson U., 1995, ApJ, 446, 741
Brandenburg A., Nordlund Å., Stein R. F., Torkelsson U., 1996a, ApJ, 458, L45
Brandenburg A., Nordlund Å., Stein R. F., Torkelsson U., 1996b, in Kato S., Inagati S., Mineshige S., Fokue J., eds, Physics of Accretion Disks. Gordon and Breach, New York, p. 285
Chen X., Abramowicz M. A., Lasota J.-P., Narayan R., Yi I., 1995, ApJ, 443, L61
Chen X., Abramowicz M. A., Lasota J.-P., 1996, ApJ, in press
Godon P., 1995, MNRAS, 277, 157
Hawley J. F., Gammie C. F., Balbus S. A., 1995, ApJ, 440, 742
Igumenshchev I. V., Chen X., Abramowicz M. A., 1996, MNRAS, 278, 236
Kato S., Abramowicz M. A., Chen X., 1996, PASJ, 48, 67
Matsumoto R., Tajima T., 1995, ApJ, 445, 767
Narayan R., 1996, in Kato S., Inagati S., Mineshige S., Fokue J., eds, Physics of Accretion Disks. Gordon and Breach, New York, p. 15
Narayan R., Honma F., Kato S., 1996, ApJ, submitted
Paczyński B., Wiita P. J., 1980, A&A, 88, 23
Różyczka M., Turner N. J., Bodenheimer P., 1995, MNRAS, 276, 1179